Stability of local quantum dissipative systems

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Abstract

Open quantum systems weakly coupled to the environment are modeled by completely positive, trace preserving semigroups of linear maps. The generators of such evolutions are called Lindbladians. For practical and theoretical reasons, it is crucial to estimate the impact that noise or errors in the generating Lindbladian can have on the evolution. In the setting of quantum many-body systems on a lattice it is natural to consider local or exponentially decaying interactions. We show that even for polynomially decaying errors in the Lindbladian, local observables and correlation functions are stable if the unperturbed Lindbladian is translationally invariant, has a unique fixed point (with no restriction on its rank) and has a mixing time which scales logarithmically with the system size. These conditions can be relaxed to the non-translationally invariant case. As a main example, we prove that classical Glauber dynamics is stable under local perturbations, including perturbations in the transition rates which may not preserve *detailed balance*.

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1 Background and previous work

The physical properties of a closed many-body quantum system are encoded in its Hamiltonian. Theoretical models of such systems typically assume some form of local structure, whereby the Hamiltonian decomposes into a sum over interactions between subsets of nearby particles. Similarly, the behavior of an open many-body quantum system is encoded in its Liouvillian. Again, this is typically assumed to have a local structure, decomposing into a sum over local Liouvillians acting on subsets of nearby particles.

Crucial to justifying such theoretical models is the question of whether their physical properties are stable under small perturbations to the local interactions. If the physical properties of a many-body Hamiltonian or Liouvillian depend sensitively on the precise mathematical form of those local terms, then it is difficult to conclude anything about physical systems, whose interactions will always deviate somewhat from theory.

Quantum information theory has motivated another perspective on many-body Hamiltonians. Rather than studying models of naturally occurring systems, it studies how many-body systems can be engineered to produce desirable behavior, such as long-term storage of information in quantum memories [8, 15, 16, 39], processing of quantum information for quantum computing [7, 9, 10, 29, 42], or simulation of other quantum systems which are computationally intractable by classical means [2, 4, 5, 22, 24]. Again, stability of these systems

under local perturbations is crucial, otherwise even tiny imperfections may destroy the desired properties. Stability in this context has been studied for self-correcting topological quantum memories, where one requires in addition, robustness against local sources of *dissipative* noise, and the relevant quantity is the minimum time needed to introduce logical errors in the system. It has been known since [1, 8] that a self-correcting quantum memory with local interactions is possible in four spatial dimensions. With the breakthrough given by the Haah code [15], it seems that such self-correcting quantum memories may be possible to engineer in three dimension.

Recently, and partially motivated by the dissipative nature of noise, this "engineering" approach has been extended to open quantum systems and many-body Liouvillians. First, theoretical, [31, 49] and then, experimental [3, 32] work, has shown that creating many-body quantum states as fixed points of engineered, dissipative Markovian evolutions can be more robust against undesirable errors and maintain coherence of quantum information for longer times. Intuitively, there is an inherent robustness in such models: the target state is independent of the initial state. If the dissipation is engineered perfectly, the system will always be driven back towards the desired state. This idea can be used to engineer dissipative systems both for storing quantum information and for carrying out computation via dissipative dynamics. However, it does not guarantee stability against errors *in the engineered Liouvillian itself*. Once again, stability against local perturbations – this time for many-body Liouvillians rather than Hamiltonians – is of crucial importance.

In the case of closed systems governed by Hamiltonians, recent breakthroughs have given rigorous mathematical justification to our intuition that the physical properties of many-body Hamiltonians are stable. Starting with [6, 30], it culminated in the work of [40] where it was shown that, under a set of mathematically well-defined and physically reasonable conditions, gapped many-body Hamiltonians are stable under perturbations to the local interactions.[1] More precisely, in the presence of *frustration-freeness*, *local topological quantum order*, and *local gaps*, the spectral gap of a Hamiltonian with (quasi) local interactions is stable against small (quasi) local perturbations (see [40] for a formal definition of these conditions). The bound on the amount of imperfection tolerated by the system comes in terms of the decay of the local gaps, the decay of the local topological order, and the strength (and decay rate) of the interactions. Furthermore, except for frustration freeness, which is a technical condition required in the proof, these conditions are in a sense tight. There exist simple counterexamples to stability if any one of the conditions is lifted.

2 Stability of open quantum systems

In this work, we study stability of many-body Liouvillians. We consider dynamics generated by rapidly decaying interactions, where the notion of rapid decay is made precise in section 3. Moreover, we restrict to Liouvillians whose local terms depend only on the subsystem on which they act, and thus are not redefined

[1] Note that, in stark contrast to traditional perturbation theory, the perturbations considered here simultaneously change *all* the local interactions by a small amount. The perturbations are therefore unbounded, and standard perturbation theory does not apply. It is the structure of local groundstates of the Hamiltonian that ensures stability.

every time we consider larger systems. We call such families of Liouvillians *uniform*.

Our main result shows that, under the above assumptions on the structure of the Liouvillian, logarithmic mixing time implies the desired stability in the dissipative setting.

However, although the result is analogous to [40], the proof and even the definition of stability in the case of Liouvillians necessarily differ substantially from the Hamiltonian case. For Hamiltonians, the relevant issue is stability of the spectral gap. Via the quasi-adiabatic technique [18, 20], this in turn implies a smooth transition between the initial and perturbed ground states, showing that both are within the same phase. Note that the existence of a smooth transition (no closing of the spectral gap in the thermodynamic limit) does not imply that both groundstates are close in norm, as the simple example $H = \sum_{i=1}^{N} |0\rangle\langle 0|_i$ vs. $H(\varepsilon) = \sum_{i=1}^{N}(|0\rangle + \varepsilon|1\rangle)(\langle 0| + \varepsilon\langle 1|)_i/(1+\varepsilon^2)$ shows.[2] It does however imply a well-behaved perturbation in the expectation value of local observables – such as order parameters – and correlation functions, which in most experimental situations are the only measurable quantities.

For Liouvillians, we are interested in a definition of stability more related to the evolution itself, which accounts at the same time for both the speed of convergence and the properties of the fixed point. Here, we consider the strongest definition of stability: we want our systems (initial and perturbed) to evolve similarly for all times and all possible initial states. Thus, not only should the speed of convergence to the fixed points be similar, the fixed points themselves should be close and so should the approach to the fixed points.

This definition is significantly stronger than stability of the spectral gap alone[3], and is more directly relevant to the applications discussed above. As in the Hamiltonian case, the analogous simple example shows that one cannot expect to attain such stability if we consider global measurements on the system. We therefore restrict our attention to local observables and few-body correlation functions. Since there are important subtleties involved in extending this stronger definition of stability to dynamics with multiple fixed points, we defer consideration of multiple fixed points to a future paper, and restrict our attention to dissipative dynamics with unique fixed points. It is important to note, however, that we do not make *any* assumption on the form of the unique fixed point. In particular, we do not assume that it is full-rank (primitivity); our results apply equally well to Liouvillians with pure fixed points[4].

A key technical ingredient in the stability proof for Hamiltonians is the quasi-adiabatic evolution technique [18, 20], which directly uses the fact that Hamiltonian dynamics is reversible. This is of course no longer true for Liouvillians, so we must use a different proof approach. We make use of the fact that evolution under a Liouvillian converges to a steady-state, together with dissipative generalizations [41] of the Lieb-Robinson bounds that are the other crucial ingredient in [40].

[2] Note that each Hamiltonian is the sum of non-interacting projections for any $\varepsilon \in \mathbb{R}$. In particular, for each ε, there is a unitary $U(\varepsilon)$ acting on a single site, such that $H(\varepsilon) = U(\varepsilon)^{\otimes N} H U^{\dagger}(\varepsilon)^{\otimes N}$.

[3] Due to the recent work in [46], it is not clear whether the spectral gap in Liouvillians is the relevant quantity for convergence questions.

[4] Pure-state fixed points are particularly relevant to quantum information applications, such as dissipative state engineering and computation.

Among systems which satisfy our assumptions, one finds classical Glauber dynamics [38]. This immediately shows that Glauber dynamics is stable against errors. To the best of our knowledge, this is new even to the classical literature. Given the importance of Glauber dynamics to sampling from the thermal distributions of classical spin systems [34, 38], we expect our results to have applications also to classical statistical mechanics.

The paper is structured as follows: After setting up notation and basic definitions in the next section, we state our main stability result in section 4 and discuss the assumptions it requires. In section 5 we prove various technical results used in the main proof, which is given in section 6. We apply these results in section 7.1 to the important example of classical Glauber dynamics, before concluding with a discussion of the results and related open questions in section 8.

3 Setup and notation

We will consider a cubic lattice[5] $\Gamma = \mathbb{Z}^D$. The ball centered at $x \in \Lambda$ of radius r will be denoted by $b_x(r)$. At each site x of the lattice we will associate one elementary quantum system with a finite dimensional Hilbert space $\mathcal{H}_x$. Then for each finite subset $\Lambda \subseteq \Gamma$, the associated Hilbert space is given by

$$\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{H}_x,$$

and the algebra of observables supported on Λ is defined by

$$\mathcal{A}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{B}(\mathcal{H}_x).$$

If $\Lambda_1 \subset \Lambda_2$, there is a natural inclusion of $\mathcal{A}_{\Lambda_1}$ in $\mathcal{A}_{\Lambda_2}$ by identifying it with $\mathcal{A}_{\Lambda_1} \otimes \mathbb{1}$. The support of an observable $O \in \mathcal{A}_\Lambda$ is the minimal set Λ' such that $O = O' \otimes \mathbb{1}$, for some $O' \in \mathcal{A}_{\Lambda'}$, and will be denoted by $\operatorname{supp} O$. We will denote by $\|\cdot\|_p$ the Schatten p-norm over $\mathcal{A}_\Lambda$. Where there is no risk of ambiguity, $\|\cdot\|$ will denote the usual operator norm (i.e. the Schatten ∞-norm).

A linear map $\mathcal{T} : \mathcal{A}_\Lambda \to \mathcal{A}_\Lambda$ will be called a *superoperator* to distinguish it from operators acting on states. The support of a superoperator $\mathcal{T}$ is the minimal set $\Lambda' \subseteq \Lambda$ such that $\mathcal{T} = \mathcal{T}' \otimes \mathbb{1}$, where $\mathcal{T}' \in \mathcal{B}(\mathcal{A}_{\Lambda'})$. A superoperator is said to be Hermiticity preserving if it maps Hermitian operators to Hermitian operators. It is said to be positive if it maps positive operators (i.e. operators of the form O^*O) to positive operators. $\mathcal{T}$ is called *completely positive* if $\mathcal{T} \otimes \mathbb{1} : \mathcal{A}_\Lambda \otimes M_n \to \mathcal{A}_\Lambda \otimes M_n$ is positive for all $n \geqslant 1$. Finally, we say that $\mathcal{T}$ is trace preserving if $\operatorname{tr} \mathcal{T}(\rho) = \operatorname{tr} \rho$ for all $\rho \in A_\Lambda$. For a general review on superoperators, see [51].

The dynamics of the system is generated by a superoperator $\mathcal{L}$, which plays a similar role to the Hamiltonian in the non-dissipative case. The evolution will be given by the one parameter semigroup $T_t = e^{t\mathcal{L}}$. The natural assumptions to make about T_t are that it is a continuous semigroup of completely positive and trace preserving maps (CPTP, sometimes also called *quantum channels*). Such

[5] We restrict to cubic lattices for the sake of exposition. The results can be extended to more general settings, replacing the lattice $\mathbb{Z}^D$ with a graph equipped with a sufficiently regular metric.

maps are always contractive, meaning that $\|T_t\|_{1\to1,cb} \leqslant 1$, where:

$$\|T\|_{1\to1,cb} = \sup_n \|T \otimes \mathbb{1}_n\|_{1\to1} = \sup_n \sup_{\substack{X\in\mathcal{A}_\Lambda\otimes M_n \\ X\neq 0}} \frac{\|T\otimes\mathbb{1}_n(X)\|_1}{\|X\|_1}. \tag{1}$$

We will also be interested in the $\|\cdot\|_{\infty\to\infty,cb}$ norm of superoperators, which is defined as follows:

$$\|T\|_{\infty\to\infty,cb} = \sup_n \|T \otimes \mathbb{1}_n\|_{\infty\to\infty} = \sup_n \sup_{\substack{X\in\mathcal{A}_\Lambda\otimes M_n \\ X\neq 0}} \frac{\|T\otimes\mathbb{1}_n(X)\|_\infty}{\|X\|_\infty}.$$

The relationship between $\|\cdot\|_{1\to1,cb}$ and $\|\cdot\|_{\infty\to\infty,cb}$ is the following:

$$\|T\|_{1\to1,cb} = \|T^*\|_{\infty\to\infty,cb},$$

where T^* is the dual of T, satisfying $\operatorname{tr} A\,T(B) = \operatorname{tr} T^*(A)\,B$. We will denote $\|\cdot\|_{\infty\to\infty,cb}$ simply by $\|\cdot\|_{cb}$ when there is no risk of confusing different completely-bounded norms.

Observation 3.1. As shown in [23], the supremum in equation (1) is reached when n is equal to the dimension of the space on which T is acting: if $T : \mathcal{M}_n \to \mathcal{M}_n$, then $\|T \otimes \mathbb{1}_n\|_{1\to1} = \|T\|_{1\to1,cb}$.

The generator $\mathcal{L}$ of the semigroup $T_t = e^{t\mathcal{L}}$, is called a *Lindbladian*. All such generators can be written in the following general form, often called the *Lindblad form* [11, 35] (see [51]):

Proposition 3.2. *$\mathcal{L}$ generates a continuous semigroup of CPTP maps if and only if it can be written in the form:*

$$\mathcal{L}(\rho) = i[\rho, H] + \sum_j L_j \rho L_j^* - \frac{1}{2}\sum_j \{L_j^* L_j, \rho\}, \tag{2}$$

where H is a Hermitian matrix, $\{L_j\}_j$ a set of matrices called the Lindblad operators, $[\cdot,\cdot]$ denotes the commutator and $\{\cdot,\cdot\}$ the anticommutator.

Since $\mathcal{L}$ is defined on a lattice, it is natural to ask that it has some form of local structure. We will say that $\mathcal{L}$ is a *local Lindbladian* if it can be written as a sum of terms, each of which is itself in Lindblad form and has bounded support:

$$\mathcal{L} = \sum_u \sum_r \mathcal{L}_u(r), \quad \operatorname{supp} \mathcal{L}_u(r) = b_u(r). \tag{3}$$

Such a decomposition is obviously always trivially possible. We are interested in the cases in which the norms of $\mathcal{L}_u(r)$ decay with r. Concretely, let us define the *strength of interaction* for a Lindbladian as the pair (J, f) given by:

$$J = \sup_{u,r} \|\mathcal{L}_u(r)\|_{1\to1,cb}, \quad f(r) = \sup_u \frac{\|\mathcal{L}_u(r)\|_{1\to1,cb}}{J}. \tag{4}$$

The behavior of $f(r)$ as r goes to infinity corresponds to various interaction regimes, listed in order of decreasing decay rate:

- finite range interaction: $f(r)$ is compactly supported;

- exponentially decaying: $f(r) \leqslant e^{-\mu r}$, for some $\mu > 0$;
- quasi-local interaction: $f(r)$ decays faster than any polynomial;
- power-law decay: $f(r) \leqslant (1+r)^{-\alpha}$, for some positive $\alpha > 0$.

As we will see later, our result will apply whenever $\mathcal{L}$ has finite range, exponentially decaying, or quasi-local interactions. It will also hold in the power-law decay regime, but we will require a lower bound on the decay exponent α, depending on the dimension of the underlying lattice. Not to overload the exposition, we will assume that $\mathcal{L}$ has finite range or exponentially decaying interactions, unless otherwise specified. The modifications needed to work with quasi-local interactions and power-law decay are presented in section 6.5. Also, we will say that functions we construct along the way are *fast-decaying*, if their decay rate is within the same decay class of $f(r)$ we are considering (or faster).

As shown in [50], from the spectral decomposition of $\mathcal{L}$ (and T_t) one can define two new CPTP maps which represent the infinite-time limit of the semigroup T_t. We will denote by T_∞ the projector onto the subspace of stationary states (fixed points), and by T_ϕ the projector onto the subspace of periodic states. They correspond, respectively, to the kernel of $\mathcal{L}$ and to the eigenspace of purely imaginary eigenvalues of $\mathcal{L}$, which we denote $\mathcal{F}_\mathcal{L}$ and $\mathcal{X}_\mathcal{L}$, respectively. Both subspaces are invariant under T_t: in particular, T_t acts as the identity over $\mathcal{F}_\mathcal{L}$, while it is a unitary operator over $\mathcal{X}_\mathcal{L}$. Note, also, that both subspaces are spanned by positive operators (i.e. density matrices) [51, Prop. 6.8, Prop. 6.12]. We will denote by $T_{\phi,t}$ the composition $T_t \circ T_\phi$.

Since we plan to exploit the local structure of $\mathcal{L}$, we will often make use of the restriction of $\mathcal{L}$ to a subset of the lattice. Given $A \subset \Lambda$, we define the *truncated*, or *localized*, generator:

$$\mathcal{L}_A = \sum_{b_u(r) \subseteq A} \mathcal{L}_u(r). \tag{5}$$

3.1 Uniform families

We are interested in how properties of dissipative dynamics scale with the size of the system. Hence, we are concerned with sequences of Lindbladians defined on lattices of increasing size, where all the Lindbladians in the sequence are from the same "family". To make this precise, we need to pin down how Lindbladians from the same family, but on different size lattices, are related to one-another. Our results will apply to very general sequences of Lindbladians, which we call *uniform families*. Before giving the precise definition, it is helpful to consider some special cases.

For local Hamiltonians on a lattice, one often considers translationally-invariant systems with various types of boundary conditions (e.g. open or periodic boundaries). There is then a natural definition of what it means to consider the same translationally-invariant Hamiltonian on different lattice sizes. Translationally-invariant Lindbladians are an important special case of a uniform family. In this special case, all the local terms in the Lindbladian that act in the "bulk" of the lattice are the same. Another way of thinking about this is to formally consider the translationally-invariant Lindbladian $\mathcal{M}$ defined on the infinite lattice $\Gamma = \mathbb{Z}^d$, and then consider each member of the family to be

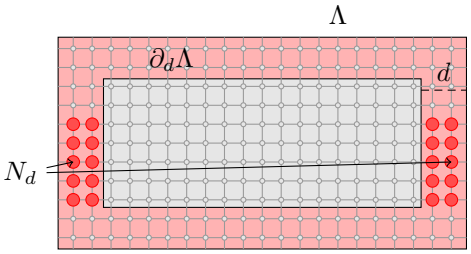

Figure 1: Partition of the lattice Λ into the bulk and the boundary of thickness d, $\partial_d\Lambda$ (see Def. 3.3). The dark red regions on the boundary correspond to the interaction term N_d coupling distant regions in Λ.

a restriction of this infinite Lindbladian to a finite sub-lattice $\Lambda \subset \Gamma$ of some particular size:

$$\mathcal{L} = \mathcal{M}_\Lambda.$$

This gives us translationally-invariant Lindbladians with *open boundary condition*. But of course, this is only one particular choice of boundary terms (in this case, no boundary terms at all). We are also interested in more general boundary conditions, such as periodic boundaries. So, in addition to the "bulk" interactions coming from $\mathcal{M}$, we allow additional terms that play the role of *boundary conditions*:

$$\mathcal{L} = \mathcal{M}_\Lambda + \mathcal{L}^{\partial\Lambda}.$$

We allow greater freedom in the boundary terms $\mathcal{L}^{\partial\Lambda}$. For one thing, they are allowed to depend on the size of the lattice Λ. But more importantly, we allow strong interactions that *cross the boundary* of Λ, coupling sites that would otherwise be far apart. For example, the case of *periodic boundary conditions* corresponds to adding interaction terms that connect opposite boundaries of Λ, as if on a torus (see Fig. 1).

Now that we have given an intuition of what a uniform family is, it is time to present the formal definition. This includes all the special cases discussed so far, but also captures much more general families of Lindbladians that are not necessarily translationally-invariant, and many other types of boundary conditions (e.g. cylindrical boundaries, or boundary terms that give the sphere topology, or terms that force fixed states on the boundary[6]).

Definition 3.3. *Given $\Lambda \subset \Gamma$, a* boundary condition *with strength (J, f) for Λ is given by a Lindbladian $\mathcal{L}^{\partial\Lambda} = \sum_{d\geqslant 1} N_d$, where*

$$\partial_d\Lambda := \{x \in \Lambda \mid \operatorname{dist}(x, \Lambda^c) \leqslant d\},$$
$$\operatorname{supp} N_d \subset \partial_d\Lambda,$$
$$\|N_d\|_{1\to 1,c.b.} \leqslant J\,|\partial_d\Lambda|\,f(d).$$

Definition 3.4. *A* uniform family *of Lindbladians $\mathcal{L}$ with strength (J, f) is given by the following:*

(i) infinite Lindbladian: a Lindbladian $\mathcal{M}$ on all of $\mathbb{Z}^D$ with strength (J, f);

[6] Or even Möbius strips, Klein bottles, and other exotic topologies.

(ii) boundary conditions: a set of boundary conditions $\mathcal{L}^{\partial\Lambda}$*, with strength* (J, f) *and* $\Lambda = b_u(L)$*, for each* $u \in \mathbb{Z}^D$ *and* $L \geqslant 0$*.*

We say that the family is translationally invariant *if* $\mathcal{M}$ *is, and moreover* $\mathcal{L}^{\partial b_u(L)}$ *is independent of* u*.*

Given a uniform family $\mathcal{L}$, we fix the following notation for evolutions defined on a subset Λ:

$$\mathcal{L}^{\Lambda} = \mathcal{M}_{\Lambda} \quad \text{“open boundary” evolution;} \tag{6}$$

$$\mathcal{L}^{\overline{\Lambda}} = \mathcal{M}_{\Lambda} + \mathcal{L}^{\partial\Lambda} \quad \text{“closed boundary” evolution,} \tag{7}$$

with the respective evolutions $T_t^{\Lambda} = \exp(t\mathcal{L}^{\Lambda})$ and $T_t^{\overline{\Lambda}} = \exp(t\mathcal{L}^{\overline{\Lambda}})$.

Observation 3.5. In the rest of the paper, we will make use of the following notation:

$$A(s) = \{x \in \Lambda \,|\, \operatorname{dist}(x, A) \leqslant s\}.$$

Since we are interested in observables whose support is not connected, we want to consider more general regions than balls: in particular, we are interested in disjoint unions of convex regions (for example, to calculate two-point correlation functions). Consider what happens to such a region $A = A_0 \sqcup A_1$ when we grow it by taking $A(s)$. When s becomes sufficiently large, $A_0(s)$ will merge with $A_1(s)$. At this point, $A(s)$ will not be a disjoint union of balls anymore. To avoid such complications, for s large enough that disjoint balls merge, we will replace $A(s)$ by the smallest ball containing it. This will not hurt us, as $|A(s)|$ will still grow asymptotically at the same rate, which will be sufficient for our purposes.

Definition 3.6. *We say that* $\mathcal{L}$ *has a unique fixed point if, for all* $\Lambda = b_u(L)$*,* $\mathcal{X}_{\mathcal{L}^{\overline{\Lambda}}} = \mathcal{F}_{\mathcal{L}^{\overline{\Lambda}}} = \{\rho_{\infty}^{\overline{\Lambda}}\}$*. In other words,* $T_{\phi}^{\overline{\Lambda}}(\rho) = T_{\infty}^{\overline{\Lambda}}(\rho) = \rho_{\infty}^{\overline{\Lambda}}$*, for all density matrices* ρ*.*

Note that if there exists a finite time t_0 such that $T_t^{\overline{\Lambda}}(\rho) > 0$ (positive definite), for all $t \geqslant t_0$ and density matrices ρ, then the evolution has a unique fixed point $\rho_{\infty} > 0$ (see [51, Thm. 6.7]).

We will drop the superscript from $T_t^{\overline{\Lambda}}$, and simply write T_t, when we consider some fixed $\Lambda \subset \Gamma$. In that case, we will refer to the number of lattice sites in Λ as the *system size*.

4 Main result

4.1 Assumptions for stability

In Hamiltonian systems, the spectral gap (the difference between the two lowest energy levels) plays a crucial role in a number of settings, from defining quantum phases and phase transitions [45] to understanding the entanglement and correlations present in the system [17, 19, 21] and analyzing its stability to perturbations [6, 40]. On the other hand, it is known that for Lindbladians, the spectral gap (in this setting, the least negative real part of the non-zero eigenvalues) alone is not sufficient to fully characterize the convergence properties

of the dissipative evolution [27, 46]. Therefore, we will instead impose a more general requirement on the convergence of the dynamics. (The dependence of this requirement on spectral properties of $\mathcal{L}$, i.e. properties depending on the eigenvalues – like the gap – and eigenvectors – like the condition number, is an active area of research.)

Definition 4.1 (Global rapid mixing)**.** *Given a one-parameter semigroup of CPTP maps S_t, define the* contraction *of S_t as the following quantity:*

$$\eta(S_t) = \frac{1}{2} \sup_{\substack{\rho \geqslant 0 \\ \operatorname{tr} \rho = 1}} \|S_t(\rho) - S_{\phi,t}(\rho)\|_1 . \tag{8}$$

We say that a uniform family of Lindbladians $\mathcal{L}$ satisfies global rapid mixing *if, for each $\Lambda = b_0(L)$,*

$$\eta(T_t^{\overline{\Lambda}}) \leqslant |\Lambda|^{\delta} e^{-t\gamma}; \tag{9}$$

for some $\gamma, \delta > 0$. We will write GRM(γ, δ) for short.

This last assumption can be restated as a logarithmic scaling with system size of the mixing time for $T_t^{\overline{\Lambda}}$.

Let us recall a result from [27].

Theorem 4.2 (Contraction for commuting Liouvillians)**.** *Let $\{\mathcal{L}_j\}_{j=0}^n$ be a set of commuting Lindbladians. Define $\mathcal{L} = \sum_j \mathcal{L}_j$ and the corresponding evolutions $T_t^j = e^{t\mathcal{L}_j}$ and $T_t = e^{t\mathcal{L}}$. Then:*

$$\eta(T_t) \leqslant \sum_j \eta(T_t^j). \tag{10}$$

In particular, consider the definition of $T_t^{\overline{\Lambda}}$ given in Observation 3.5 for $\Lambda \subset \Gamma$ being a of disjoint union of balls. Then the previous Theorem implies that, if $\mathcal{L}$ is translation invariant and it satisfies global rapid mixing, it holds that

$$\eta(T_t^{\overline{\Lambda}}) \leqslant |\Lambda|^{\delta} e^{-t\gamma}. \tag{11}$$

The result and the proof we are going to present will only make use of this property instead of global rapid mixing. For non translational invariant systems, one could in principle check the scaling of $\eta(T_t^{\overline{\Lambda}})$ for all regions Λ of this particular shape, and verify equation (11) without having to require additional symmetries of the interactions.

4.2 Stability

With the required assumptions laid out, we can now state our main result.

Theorem 4.3. *Let $\mathcal{L}$ be a uniform family of local Lindbladians with a unique fixed point, satisfying equation (11).*

Let $E = \sum_u \sum_r E_u(r)$ be a sum of super-operators, which we will call the perturbation, such that $E_u(r)$ is supported on $b_u(r)$ (with respect to the geometry of $b_0(L)$ chosen in Definition 3.4) and $\|E_u^\star(r)\|_{cb} \leqslant \varepsilon\, e(r)$, where $\varepsilon > 0$ is a constant (the "strength" of the perturbation) and $e(r)$ is a fast-decaying function.

Consider the perturbed evolution

$$S_t = \exp t(\mathcal{L}^{\overline{\Lambda}} + E_\Lambda).$$

and suppose that E satisfies the following assumptions:

(i) $E_u^*(r)[\mathbb{1}] = 0$.

(ii) S_t *is a contraction for each* $t \geqslant 0$.

For an observable O_A supported on $A \subset \Lambda$, let $O_0(t) = T_t^* O_A$ and $O_1(t) = S_t^* O_A$. Then, we have that for all $t \geqslant 0$:

$$\|O_0(t) - O_1(t)\| \leqslant \varepsilon\, p(|A|)\, \|O_A\|\,, \tag{12}$$

where $p(|A|)$ is independent of Λ and t, and is bounded by a polynomial in $|A|$.

Observation 4.4. The assumptions (i)-(ii) on the perturbation E are satisfied whenever $\mathcal{L}_u(r)+E_u(r)$ is a Lindbladian, but there are more general perturbations which are covered by the theorem.

Observation 4.5. Since we are free to choose an O_A with support on two non connected regions, we can apply theorem 6.9 to two-point correlation functions (or more generally k-point correlation functions, for fixed k) and still obtain that the error introduced by the perturbation depends linearly on the strength of the perturbation (and not on its global norm).

A set of tools already applied in the setting of classical Markov chains [12–14, 38], and recently generalized to quantum dissipative systems [27], are the so-called *Logarithmic Sobolev inequalities* (in short, log-Sobolev inequalities). Introduced in a different setting to study hypercontractivity of semigroups [28], they provide the right asymptotic regime needed to satisfy the global rapid mixing hypothesis: in fact, the existence of a system size independent log-Sobolev constant implies a logarithmic scaling of the mixing time, which is exactly what is required in Definition 4.1. Without going into the technical details of log-Sobolev inequalities, we summarize this fact in the following Corollary.

Corollary 4.6. *Let $\mathcal{L}$ belong to a uniform, translational invariant family of Lindbladians, having a unique fixed point for each system size. If $\mathcal{L}$ satisfies the log-Sobolev inequality with a system-size independent constant, then the system is stable, in the sense of theorem 4.3.*

A straightforward consequence of the techniques we use to prove stability, is that the fixed points of local Lindbladians satisfying the assumptions of theorem 4.3, necessarily have fast decay of correlations.[7]

Theorem 4.7. *Let $\mathcal{L}$ be a family Lindbladians satisfying the conditions of theorem 4.3. Fix $\Gamma \subset \Lambda$, and consider ρ_∞ the unique fixed point of $\mathcal{L}^{\overline{\Gamma}}$. Let A, $B \subset \Gamma$ and call d_{AB} the distance between them. Let us denote the mutual information of ρ_∞ between A and B as $I(A:B)$ (see definition 6.6). Then we have that:*

$$I(A:B) \leqslant p(|A| + |B|)\, g(d_{AB}),$$

where $g(\cdot)$ is a fast-decaying function and $p(\cdot)$ is bounded by a polynomial.

[7] As we were writing up this manuscript, a result similar to Theorem 4.7 was announced by Kastoryano et al. [25, 26].

4.3 Local observables vs. global observables

The bound in equation (12) has a scaling on the size of the support of the observable O_A. Although the dependence is polynomial, for observables with large support the result is not useful. Still, in most realistic experiments, we are interested in the behavior of observables with fixed support and low-degree correlation functions, making the above result widely applicable. Nonetheless, one might ask more generally for a system-size independent bound on:

$$\sup_{\rho} \|T_\infty(\rho) - S_\infty(\rho)\|_1, \tag{13}$$

where S_∞ is the fixed-point projector for the evolution of the perturbed Lindbladian. However, this is not possible; the limitation to local observables is in some sense strict. There is no hope of finding such a bound for global observables, as the following simple example shows.[8]

Example 4.8. Consider N independent amplitude damping processes, with uniform rate γ (which we can suppose w.l.o.g. equal to 1). This Lindbladian can be written as

$$\mathcal{L}_N = \sum_{k=1}^{N} \mathbb{1}_{1\dots k-1} \otimes \mathcal{L}_1 \otimes \mathbb{1}_{k+1\dots N},$$

where

$$\mathcal{L}_1(\rho) = |0\rangle\langle 1|\,\rho\,|1\rangle\langle 0| - \frac{1}{2}\{\rho, |1\rangle\langle 1|\}$$

is an amplitude damping process on a single qubit, describing the decay of the excited state $|1\rangle$ into $|0\rangle$ at a constant rate $\gamma = 1$. This Lindabladian has gap $1/2$ and $e^{t\mathcal{L}_N} = (e^{t\mathcal{L}_1})^{\otimes N}$ has mixing time of order $O(\log N)$ [27, Sec. V. C.]. The fixed point is the pure state $|0\dots 0\rangle\langle 0\dots 0|$.

Now consider $\mathcal{L}_1^\varepsilon$, a rotation of $\mathcal{L}_1$, which prepares the state $|\alpha_0\rangle = \sqrt{1-\varepsilon^2}\,|0\rangle + \varepsilon\,|1\rangle$. We have $\|\mathcal{L}_1 - \mathcal{L}_1^\varepsilon\|_{1\to 1} = O(\varepsilon)$, but the new fixed point $|\alpha_0\rangle\langle\alpha_0|^{\otimes N}$ is almost orthogonal to the original one, since the overlap between the two is

$$\langle 0\dots 0|\alpha_0\dots\alpha_0\rangle = \langle 0|\alpha_0\rangle^N = (1-\varepsilon^2)^{N/2} \sim e^{-N\varepsilon^2/2} \to 0 \text{ as } N\to\infty.$$

This shows that, in general, there is no good bound on (13) (note that we have $\|\,|0\dots 0\rangle\langle 0\dots 0| - |a_0\dots\alpha_0\rangle\langle a_0\dots\alpha_0|\,\|_1 \geqslant 1 - |\langle 0\dots 0|\alpha_0\dots\alpha_0\rangle|^2$) and that the dependence on the support of the observable in equation (12) cannot be improved: to see this consider the observable $O_r = |0\dots 0\rangle\langle 0\dots 0|_{1\dots r}$ acting on $r \leqslant N$ spins. O_r has norm one, and

$$O_\infty := \lim_{t\to\infty} T_t^* O_r = \mathbb{1}, \quad O_\infty^\varepsilon := \lim_{t\to\infty} T_t^{\varepsilon\,*} O_r = \langle 0|\alpha_0\rangle^{2r}\,\mathbb{1} = (1-\varepsilon^2)^r\,\mathbb{1};$$

and consequently we have a

$$\|O_\infty - O_\infty^\varepsilon\| = 1 - (1-\varepsilon^2)^r = r\varepsilon^2 + o(\varepsilon^2).$$

This implies that any upper bound to $\|O_\infty - O_\infty^\varepsilon\|$ has to be at least linear in r, which is the size of the support of O_r.

[8] Indeed, all global stability results for quantum Linbladians we are aware of have a dependency on the total Hilbert space dimension [26, 47].

4.4 Do we need all the assumptions?

It is reasonable to ask if the assumptions of theorem 4.3 are all necessary. We have just shown that we must necessarily consider local observables if we are to have meaningful bounds, but what about the other conditions? We will now present three examples, each consisting in a family of Lindbladiands with periodic boundary conditions, such that, in order:

1. it is uniform and translational invariant, satisfies global rapid mixing but does not have a unique fixed point;
2. has a unique fixed point, but it is not uniform and fails to satisfy global rapid mixing;
3. (presented in Appendix A) has a unique fixed point, satisfies global rapid mixing but it is not uniform.

All these systems will be shown to be unstable.

Example 4.9. Consider a 1D chain composed of N 4-level systems, with an independent Lindbladian acting on each site, having the following Lindblad operators

$$L_1 = |0\rangle\langle 1|\,, \quad L_2 = |0\rangle\langle 3|\,, \quad L_3 = |2\rangle\langle 1|\,, \quad L_4 = |2\rangle\langle 3|\,,$$

and call

$$\mathcal{L}_0(\rho) = \sum_{i=1}^{4} L_i \rho L_i^* - \frac{1}{2}\{\rho, L_i^* L_i\}.$$

The global Lindbladian $\mathcal{L}_N$ is given by applying $\mathcal{L}_0$ independently on each site $k = 1 \dots N$:

$$\mathcal{L}_N = \sum_{k=1}^{N} \mathbb{1}_{1,\dots,k-1} \otimes \mathcal{L}_0 \otimes \mathbb{1}_{k+1,\dots,N}.$$

Then we have that

$$\mathcal{L}_0(|i\rangle\langle j|) = \begin{cases} |0\rangle\langle 0| + |2\rangle\langle 2| - 2\,|i\rangle\langle j| & \text{if } i = j,\ i,j \in \{1,3\} \\ 0 & \text{if } i = j,\ i,j \in \{0,2\} \\ -[\chi_{\{1,3\}}(i) + \chi_{\{1,3\}}(j)]\,|i\rangle\langle j| & \text{if } i \neq j. \end{cases}$$

Diagonal states of the form $|i\rangle\langle i|$ evolve according to the classical Markov process embedded in the Lindbladian, while off-diagonal elements $|i\rangle\langle j|$ evolve as

$$T_t(|i\rangle\langle j|) = \exp(-t[\chi_{\{1,3\}}(i) + \chi_{\{1,3\}}(j)])\,|i\rangle\langle j|\,;$$

where $\chi_{\{1,3\}}$ denotes the indicator function of the set $\{1,3\}$. This implies that $\mathcal{F}_{\mathcal{L}_0} = \mathrm{span}\{|0\rangle\langle 0|\,, |2\rangle\langle 2|\,, |0\rangle\langle 2|\,, |2\rangle\langle 0|\}$. Since $\mathcal{L}_0$ has gap equal to 1, Theorem 4.2 implies that $\mathcal{L}_N$ satisfies global rapid mixing. $\mathcal{L}_N$ forms a uniform family, but it does not satisfy the unique fixed point condition.

Consider now the following additional Lindbladian

$$\mathcal{E}_0(\rho) = \frac{2}{N}\left[|0\rangle\langle 2|\,\rho\,|2\rangle\langle 0| - \frac{1}{2}\{\rho, |2\rangle\langle 2|\}\right].$$

Then, we have:

$$(\mathcal{L}_0+\mathcal{E}_0)(|i\rangle\langle j|)=\begin{cases}|0\rangle\langle 0|+|2\rangle\langle 2|-2\,|i\rangle\langle j| & \text{if } i=j,\ i,j\in\{1,3\}\\ \frac{2}{N}(|0\rangle\langle 0|-|i\rangle\langle j|) & \text{if } i=j=2\\ 0 & \text{if } i=j=0\\ -\left[\chi_{\{1,3\}}(i)+\chi_{\{1,3\}}(j)+\frac{\chi_{\{i,j\}}(2)}{N}\right]|i\rangle\langle j| & \text{if } i\neq j.\end{cases}$$

Again, this implies that $\mathcal{F}_{\mathcal{L}_0+\mathcal{E}_0}=\{|0\rangle\langle 0|\}$. Consequently $\mathcal{L}_N+\mathcal{E}_N$ has a unique fixed point. It is not a uniform family, and it does not satisfy global rapid mixing, as it is not even globally gapped. To see this, note that for $\sigma=|200\ldots0\rangle\langle 200\ldots0|-|020\ldots0\rangle\langle 020\ldots0|$:

$$(\mathcal{L}_N+\mathcal{E}_N)(\sigma)=-\frac{2}{N}\,\sigma.$$

Analogously, $\mathcal{L}_N+\mathcal{E}_N^*$ satisfies the same conditions as $\mathcal{L}_N+\mathcal{E}_N$, but the unique fixed point is now $|2\ldots2\rangle\langle 2\ldots2|$.

All three systems described above are unstable, since we can transform one into the other by applying a perturbation of order $O(1/N)$, yet the fixed points of $\mathcal{L}_N+\mathcal{E}_N$ and $\mathcal{L}_N+\mathcal{E}_N^*$ are locally orthogonal (while $\mathcal{L}_N$ has both of them as fixed points).

5 Toolbox for the proof

Before presenting the proof of Theorem 4.3, we will need to introduce some useful tools. We will present them in full generality, including the case of power-law decay of interactions, and we will not restrict the exposition (as we did in the rest of this manuscript) to exponentially decaying interactions.

5.1 Lieb-Robinson bounds for Lindbladian evolution

We first recall a generalization of Lieb-Robinson bounds to non-Hamiltonian evolution, due to [44] and [41], which we will then use to derive a number of useful tools that allow us to approximate the support of an evolving observable with a finite set which grows linearly in time. The following condition is sufficient for the bounds to hold.

Assumption 5.1 (Lieb-Robinson condition)**.** *Let $\mathcal{L}=\sum_{u,r}\mathcal{L}_u(r)$ be a local Lindbladian. There exist some positive μ and v such that*

$$\sup_{u\in\Lambda}\sum_{x\in\Lambda}\sum_{r\geqslant\operatorname{dist}(u,x)}\|\mathcal{L}_x(r)\|_{1\to1,cb}\,|b_x(r)|\,\nu_\mu(r)\leqslant\frac{v}{2}<\infty; \tag{14}$$

where $\nu_\mu(\cdot)$ is either of the following

$$\nu_\mu(r)=e^{\mu r}, \tag{LR-1}$$
$$\nu_\mu(r)=(1+r)^\mu. \tag{LR-2}$$

Note that both functions are submultiplicative, in the sense that $\nu_\mu(r+s)\leqslant\nu_\mu(r)\nu_\mu(s)$. The constant v is called the Lieb-Robinson speed *(or velocity) of $\mathcal{L}$, while the reciprocal function $\nu_\mu^{-1}(r)=1/\nu_\mu(r)$ is called the* Lieb-Robinson decay *of $\mathcal{L}$.*

Note that if $\mathcal{L}$ has interaction strength (J, f), then equation (14) can be replaced with

$$\sum_{r\geqslant 0} |b_0(r) \setminus b_0(r-1)| \sum_{\delta\geqslant r} f(\delta)\nu_\mu(\delta)\,|b_0(\delta)| \leqslant \frac{v}{2J} < \infty. \tag{15}$$

This implies that, if equation (15) is verified, different Lindbladians whose strengths can be uniformly bounded will have uniform Lieb-Robinson velocities.

Observation 5.2. Condition (LR-1) is verified if $\mathcal{L}$ has finite-range or exponentially decaying interactions, while condition (LR-2) is verified if $\mathcal{L}$ has quasi-local interactions. If $\mathcal{L}$ has power-law decay of interaction with exponent α, then condition (LR-2) is verified whenever $\alpha > 2D+1$ (by choosing $\mu < \alpha - (2D+1)$).

Theorem 5.3 (Lieb-Robinson bound)**.** *Suppose $\mathcal{L}$ is a local Lindbladian verifying Assumption 5.1. Let O_X be an observable supported on $X \subset \Lambda$, and denote by $O_X(t) = T_t^*(O_X)$ its evolution under $\mathcal{L}$. Let $K : \mathcal{A}_Y \to \mathcal{A}_Y$ be a super-operator supported on $Y \subset \Lambda$ which vanishes on $\mathbb{1}$. Then, the following bound holds [41, 44]:*

$$\|K(O(t))\| \leqslant \|K\|_{\infty\to\infty,cb}\,\|O_X\|\,C(X,Y)\frac{\left(e^{vt}-1\right)}{\nu_\mu(\operatorname{dist}(X,Y))}, \tag{16}$$

where $C(X,Y) = \min(|X|,|Y|)$.

From now on, we will only consider Lindbladians which verify equation (15) with either of the two possible assumptions on $\nu_\mu(\cdot)$.

Lemma 5.4 (Comparing different dynamics)**.** *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two local Lindbladians, and suppose $\mathcal{L}_2$ has Lieb-Robinson speed and decay bounded by v and ν_μ^{-1}. Consider O_X an operator supported on $X \subset \Lambda$, and call $O_i(t)$ its evolution under $\mathcal{L}_i$, $i = 1,2$. Suppose that $\mathcal{L}_1 - \mathcal{L}_2 = \sum_{r\geqslant 0} M_r$, where M_r is a superoperator supported on Y_r which vanishes on $\mathbb{1}$, and $\operatorname{dist}(X, Y_r) \geqslant r$. Then, it holds that:*

$$\|O_1(t) - O_2(t)\| \leqslant \|O_X\|\,|X|\,\frac{e^{vt} - vt - 1}{v}\sum_{r=0}^{\infty}\|M_r\|_{1\to 1,cb}\,\nu_\beta^{-1}(r). \tag{17}$$

Proof. Let $h(t) = O_1(t) - O_2(t)$. Calculating its derivative, we obtain

$$h'(t) = \mathcal{L}_1^* O_1(t) - \mathcal{L}_2^* O_2(t) = \mathcal{L}_1^* h(t) + (\mathcal{L}_1^* - \mathcal{L}_2^*)O_2(t).$$

Since $h(0) = 0$, this differential equation for $h(t)$ has solution

$$h(t) = O_1(t) - O_2(t) = \int_0^t e^{(t-s)\mathcal{L}_1^*}(\mathcal{L}_1^* - \mathcal{L}_2^*)O_2(s)\,\mathrm{d}s = \sum_{r\geqslant 0}\int_0^t e^{(t-s)\mathcal{L}_1^*} M_r^* O_2(s)\,\mathrm{d}s,$$

giving us a useful integral representation for $O_1(t) - O_2(t)$. From this, we obtain the estimate

$$\|O_1(t) - O_2(t)\| \leqslant \sum_{r\geqslant 0}\int_0^t \|M_r^* O_2(s)\|\,\mathrm{d}s,$$

where we have used the fact that $e^{t\mathcal{L}_1^*}$ is a contraction with respect to $\|\cdot\|_\infty$ for each $t \geqslant 0$.

We can now apply the Lieb-Robinson bound (equation (16)) to each of the terms in the sum in the previous estimate, to obtain:

$$\|O_1(t) - O_2(t)\| \leqslant \sum_{r \geqslant 0} \|M_r\|_{1\to 1,cb} \|O_X\| \, C(X, Y_r) \nu_\mu^{-1}(\operatorname{dist}(X, Y_r)) \int_0^t (e^{vs} - 1) \, \mathrm{d}s,$$

which implies the claimed bound. □

A particular case of use for the previous Lemma, is when $\mathcal{L}_2$ is a restriction of $\mathcal{L}_1$ onto a smaller region. Since this setting is so common, and has a lot of interest, we will state a specialized result for it.

Lemma 5.5 (Localizing the evolution)**.** *Let O_A be an observable supported on a finite $A \subset \Lambda$. Denote by $O_A(t) = T_t^*(O_A)$ its evolution under a local Lindbladian $\mathcal{L}$ with strength (J, f). Given $r > 0$, denote by $O_r(t)$ its evolution under the localized Lindbladian $\mathcal{L}_{A(r)}$.*

Then, the following bound holds:

$$\|O_A(t) - O_r(t)\| \leqslant \|O_A\| \, |A| \, J \frac{e^{vt} - 1 - vt}{v} \nu_\beta^{-1}(r), \tag{18}$$

where $\nu_\beta^{-1}(r)$ decays exponentially if $\mathcal{L}$ verifies condition (LR-1), while decays as $(1+r)^{-\beta}$ if $\mathcal{L}$ verifies condition (LR-2). In this case, if we call α the decaying rate of $\mathcal{L}$, then β is given by:

$$\beta = \begin{cases} \alpha - 3D & \text{if } \alpha \geqslant 5D - 1; \\ \frac{1}{2}(\alpha - D - 1) & \text{if } \alpha \leqslant 5D - 1. \end{cases}$$

Proof. First, let us decompose $\mathcal{L} - \mathcal{L}_{A(r)}$ as a telescopic sum

$$\mathcal{L} - \mathcal{L}_{A(r)} = \sum_{l \geqslant r} \mathcal{L}_{A(l+1)} - \mathcal{L}_{A(l)}.$$

Since each element in the sum is the difference between restrictions on different subsets of the same global Lindbladian, it is easy to explicitly write their difference

$$\mathcal{L}_{A(l+1)} - \mathcal{L}_{A(l)} = \sum_{\delta=0}^{l+1} \sum_{\operatorname{dist}(u,A)=\delta} \mathcal{L}_u(l + 1 - \delta).$$

Let us group the terms in the sum by their distance from A: call

$$d = \operatorname{dist}(A, b_u(l + 1 - \delta)) = \max\{0, 2\delta - l - 1\}$$

and

$$M_0 = \sum_{l \geqslant r} \sum_{\delta=0}^{\frac{l+1}{2}} \sum_{\operatorname{dist}(u,A)=\delta} \mathcal{L}_u(l + 1 - \delta); \tag{19}$$

$$M_d = \sum_{l \geqslant r} \sum_{\substack{\operatorname{dist}(u,A)=\delta \\ \delta = \frac{l+1+d}{2}}} \mathcal{L}_u(l + 1 - \delta). \tag{20}$$

in such a that that:

$$\sum_{d\geqslant 0} M_d = \mathcal{L} - \mathcal{L}_{A(r)}; \qquad \operatorname{dist}(A, \operatorname{supp} M_d) = d.$$

We can then apply Lemma 5.4, and obtain:

$$\|O_A(t) - O_r(t)\| \leqslant \|O_A\|\,|A|\,J\frac{e^{vt}-1-vt}{v}\zeta(r);$$

where, by denoting $q(l) = |A(l) \setminus A(l-1)|$, $\zeta(r)$ is the following:

$$\zeta(r) = \frac{1}{J}\sum_{d\geqslant 0}\|M_d\|_{1\to 1,cb}\,\nu_\mu^{-1}(d) \leqslant$$
$$\sum_{l\geqslant r}\sum_{\delta=0}^{\frac{l+1}{2}} q(\delta)f(l+1-\delta) + \sum_{\delta=\frac{l+1}{2}}^{l+1} q(\delta)f(l+1-\delta)\nu_\mu^{-1}(2\delta-l-1). \quad (21)$$

If $\delta \geqslant (l+1)/2$, since $\nu_\mu(\cdot)$ is submultiplicative, then:

$$\nu_\mu(\delta) \leqslant \nu_\mu(l+1-\delta)\nu_\mu(2\delta-l-1).$$

Otherwise, since $\nu_\mu(\cdot)$ is increasing, we have that $\nu_\mu(\delta) \leqslant \nu_\mu(l+1-\delta)$. Plugging these inequalities in the above sum, we get:

$$\zeta(r) \leqslant \sum_{l\geqslant r}\sum_{\delta=0}^{l+1}\left[q(\delta)\nu_\mu^{-1}(\delta)\right]\left[f(l+1-\delta)\nu_\mu(l+1-\delta)\right].$$

Since f verifies equation (15), which in particular implies

$$\sum_{\delta\geqslant 0} f(\delta)\nu_\mu(\delta)\,|b_0(\delta)| < \infty,$$

then the sequence $f(\delta)\nu_\mu(\delta)$ is decreasing. We distinguish two cases: If ν_μ is of the type (LR-1), then the decay of $f(\delta)\nu_\mu(\delta)$ is exponential. Since $q(\delta)$ grows polynomially, $q(\delta)\nu_\mu^{-1}(\delta)$ is exponentially decaying. Then, the convolution of the two sequences, which is exactly:

$$\sum_{\delta=0}^{l+1}\left[q(\delta)\nu_\mu^{-1}(\delta)\right]\left[f(l+1-\delta)\nu_\mu(l+1-\delta)\right]$$

is exponentially decaying too, which implies an exponential decay rate for $\zeta(r)$. Thus, there exists some $\beta > 0$ such that $\zeta(r) \leqslant \nu_\beta^{-1}(r)$, and this concludes the proof for the case of exponential decay. Let us suppose now that ν_μ is of type (LR-2). Then, $f(\delta)\nu_\mu(\delta)$ decays as $(1+\delta)^{\mu-\alpha}$, while $q(\delta)\nu_\mu^{-1}(\delta)$ decays as $(1+\delta)^{D-1-\mu}$. This implies that their convolution decays as $(1+l)^{-\min(\alpha-\mu,\mu-D+1)}$[9] and thus

$$\zeta(r) \leqslant c(1+r)^{-\min(\alpha-\mu-1,\mu-D)} = \nu_\beta^{-1}(r).$$

Recalling that condition (LR-2) requires $\mu < \alpha - (2D+1)$, a simple calculation shows that the above decay rate is maximized for

$$\mu < \min\left(\alpha - 2D - 1, \frac{\alpha + D - 1}{2}\right),$$

which gives the claimed formula for β. □

[9] Consider two positive decreasing sequences (x_n) and (y_n). Since $0 < p < 1$ implies that $(x+y)^p \leqslant x^p + y^p$, it holds that $(x * y)_n^p \leqslant \sum_k x_k^p y_{n-k}^p = (x^p * y^p)_n$.

Another specialization of lemma 5.4, similar in spirit to the one just presented, is when we compare the evolution of local observables under $\mathcal{L}^{A(r)}$ and $\mathcal{L}^{\overline{A(r)}}$, as defined in definition 3.4.

Lemma 5.6. *Let O_A an observable supported on $A \subset \Lambda$. Given $r > 0$, it holds that*

$$\left\| T_t^{\overline{A(r)}*} O_A - T_t^{A(r)*} O_A \right\| \leqslant \|O_A\| \, |A| \, \frac{e^{vt} - 1 - vt}{v} \nu_\beta^{-1}(r). \tag{22}$$

Proof. Without loss of generality, we consider the case of $A(r)$ being a convex set. Let $k = \frac{1}{2} \operatorname{diam} A$.

By construction, $\mathcal{L}^{\overline{A(r)}} - \mathcal{L}^{A(r)} = \mathcal{L}^{\partial A(r)}$, and

$$\mathcal{L}^{\partial A(r)} = \sum_{d \geqslant 1} N_d,$$

where each N_d act on sites that are closer than d to the border of $A(r)$.

We want now to group these terms by their distance from A. Let

$$\begin{aligned} M_0 &= \sum_{i=0}^{k} N_{r+1+i}, \\ M_j &= N_{r+1-j}, \quad j = 1 \dots r; \end{aligned}$$

then we have that $\operatorname{dist}(A, \operatorname{supp} M_j) = j$. By applying 5.4 we then have that

$$\left\| T_t^{\overline{A(r)}*} O_A - T_t^{A(r)*} O_A \right\| \leqslant \|O_A\| \, |A| \, \frac{e^{vt} - 1 - vt}{v} \sum_{j=0}^{r} \|M_j\|_{1\to 1, c.b.} \, \nu_\mu^{-1}(j).$$

We are left to prove that the sum appearing in the r.h.s. is fast-decaying in r. From Definition 3.4 it follows that for $j > 0$:

$$\|M_j\|_{1\to 1, c.b.} \leqslant J \, |\partial_{r-j} A(r)| \, f(r+1-j) = J \, |A(r) \setminus A(j)| \, f(r+1-j),$$

while for $j = 0$:

$$\|M_0\|_{1\to 1, c.b.} \leqslant \sum_{i=0}^{k} J \, |\partial_{r+i} A(r)| \, f(r+1-i).$$

By calling $h(a, b) = |b_0(a) \setminus b_0(b)|$, we have that:

$$\sum_{j=0}^{r} \|M_j\|_{1\to 1, c.b.} \, \nu_\mu^{-1}(j) \leqslant$$

$$J \left[\sum_{i=0}^{k} h(r+k, k-i) f(r+1-i) + \sum_{j=1}^{r} h(r+k, k+j) f(r+1-j) \nu_\mu^{-1}(j) \right] := J\zeta(r).$$

By an argument similar to the one in Lemma 5.5, we can show that $\zeta(r)$ is fast-decaying. Indeed, $h(r+k, k-i) f(r+1-i)$ scales asymptotically as $r^D f(r)$, while $h(r+k, k+j) f(r+1-j)$ as $(r-j)^D f(r+1-j)$. If $\mathcal{L}$ verifies (LR-1), then $\zeta(r)$ will be exponentially decaying, with rate $\min(\alpha, \mu) - 1 = \mu - 1$.

If otherwise $\mathcal{L}$ verifies (LR-2), then $\zeta(r)$ has a polynomial decay, with rate $\min(\alpha - D, \mu) - 1 = \mu - 1$. In both cases than:

$$\zeta(r) \leqslant \nu_{\mu-1}^{-1}(r).$$

Notice that the constant β defined in Lemma 5.5 is smaller than $\mu - 1$. □

5.2 Local rapid mixing

Local rapid mixing is the intermediate step we are going through in order to prove theorem 4.3.

Definition 5.7 (Local rapid mixing)**.** *Take $A \subset \Lambda$, and define the* contraction of T_t relative to A *as*

$$\begin{aligned}\eta^A(T_t) &= \sup_{\substack{\rho\geqslant 0\\ \operatorname{tr}\rho=1}} \sup_{\substack{O_A\in\mathcal{A}_A\\ \|O_A\|=1}} \|O_A\left[T_t(\rho)-T_{\phi,t}(\rho)\right]\|_1 \\ &= \sup_{\substack{\rho\geqslant 0\\ \operatorname{tr}\rho=1}} \|\operatorname{tr}_{A^c}\left[T_t(\rho)-T_{\phi,t}(\rho)\right]\|_1 . \end{aligned} \tag{23}$$

We say that $\mathcal{L}$ verifies local rapid mixing *if, for each $A \subset \Lambda$, we have that*

$$\eta^A(T_t) \leqslant k(|A|)e^{-\gamma t}, \tag{24}$$

where $k(r)$ grows polynomially in r and $\gamma > 0$, with all the constants appearing above are independent of the system size.

Observation 5.8. It follows from the definition that $\eta^A(T_t) \leqslant \eta^B(T_t)$ whenever $A \subset B$. In particular, $\eta^A(T_t) \leqslant \eta(T_t)$.

Note that, in contrast with definition 4.1, the quantity $\eta^A(T_t)$ depends on the evolution *on the whole system* Λ, and not just on the subset A. Thus local rapid mixing is a very strong condition: the term $k(r)$ appearing in equation (24) only depends on the support of A, and so the reduced mixing time (i.e. the time it takes for the reduced density matrix on the subset A to converge) is independent of system size.

Example 5.9. A simple dissipative system verifying definition 5.7 is the tensor product of amplitude damping channels acting (with the same rate) on different qubits. Note that, though it might seem a trivial example, there are interesting dissipative systems of this form: among others, dissipative preparation of graph states [27] can be brought into this form by a non-local unitary rotation (which of course does not change the convergence rates).

6 Proof of main result

The proof of theorem 4.3 will be done in three steps. First, we will show that the hypotheses of theorem 4.3 imply that the fixed points of $\mathcal{L}^{\overline{\Lambda}}$ for different Λ are locally indistinguishable. Secondary, we will prove that the hypotheses of theorem 4.3 imply local rapid mixing. The third step, independent of the first two, will consist in proving that local rapid mixing and uniqueness of the fixed point imply the desired stability result.

6.1 Step 1: closeness of the fixed points

Topological quantum order (TQO), namely the property of certain orthogonal quantum states to be locally indistinguishable from each other, is a widely studied property of groundstate subspaces in the Hamiltonian setting. In the dissipative setting on the other hand, where the concept of groundstates is no

longer applicable, one may define the analogous concept for periodic states of Lindbladians. Below we describe the concept of Local Topological Quantum Order (LTQO) [40], which extends the concept of TQO to the invariant subspace (periodic states) of local restrictions of the global Lindbladian.

Please note that, in order to prove the stability result, we did not require any extra assumption like LTQO or frustration-freeness, as in the Hamiltonian proof. We will show in this section that indeed the rapid mixing hypotesis is sufficient to prove LTQO and a property similar to frustation freeness. These properties will play a role in the proof of stability, via Lemma 6.4.

Definition 6.1 (Local Topological Quantum Order (LTQO))**.** *Consider a Lindbladian $\mathcal{L}$. Take a convex set $A \subset \Lambda$ and let $A(\ell) = \{x \in \Lambda | \operatorname{dist}(x, A) \leqslant \ell\}$. Given two states $\rho_i \in \mathcal{X}_{\mathcal{L}_{A(\ell)}}$, $i = 1, 2$, consider their reduced density matrices on A:*

$$\rho_i^A = \operatorname{tr}_{A(\ell)\setminus A} \rho_i, \quad i = 1, 2.$$

We say that $\mathcal{L}$ has local topological quantum order *(LTQO) if for each $\ell \geqslant 0$:*

$$\left\| \rho_1^A - \rho_2^A \right\|_1 \leqslant p(|A|)\, \Delta_0(\ell), \tag{25}$$

where $\Delta_0(\ell)$ is a fast-decaying function, and $p(\cdot)$ is a polynomial.

As a first step in the proof, we will show that the hyphotheses of Theorem 4.3 imply that the fixed point of T_t, the fixed point of $T_t^{\overline{A}}$ and the periodic points of T_t^A are difficult to distinguish locally, in the same spirit as the LTQO condition.

Lemma 6.2. *Let $\mathcal{L}$ be a uniform family satisfying condition (11), and suppose each $\mathcal{L}^{\overline{A}}$ has a unique fixed point and no other periodic points. Let O_A be an observable supported on $A \subset \Lambda$, ρ a periodic point of $T_t^{A(s)}$ and ρ_∞^s the unique fixed point of $T_t^{\overline{A(s)}}$. Then, we have*

$$|\operatorname{tr} O_A(\rho - \rho_\infty^s)| \leqslant |A|^\delta\, \Delta_0(s) \tag{26}$$

where $\Delta_0(s)$ is a fast-decaying function.

Proof. Fix a $t \geqslant 0$, to be determined later. Since $T_t^{A(s)}$ act on its space of periodic points as a unitary evolution, there exists a $\rho' = (T_t^{A(s)})^{-1}(\rho)$, which is still a periodic point of $\mathcal{L}^{A(s)}$. Then, by the triangle inequality, we have:

$$|\operatorname{tr} O_A(\rho - \rho_\infty^s)| \leqslant \left|\operatorname{tr} O_A(T_t^{A(s)}(\rho') - T_t^{\overline{A(s)}}(\rho'))\right| + \left|\operatorname{tr} O_A(T_t^{\overline{A(s)}}(\rho') - \rho_\infty^s)\right|.$$

The first term is bounded by lemma 5.6, while the second one using the global rapid mixing hypothesis on $T_t^{\overline{A(s)}}$. Putting the two bounds together, we obtain

$$|\operatorname{tr} O_A(\rho - \rho_\infty^s)| \leqslant \|O_A\|\, |A|\, \frac{J}{v} e^{vt} \nu_\beta^{-1}(s) + |A(s)|^\delta\, e^{-\gamma t}.$$

By choosing $t = t(s)$ such that $e^{vt}\nu_\beta^{-1}(s)$ is fast decaying, we have that

$$\Delta_0(s) = \frac{J}{v} e^{vt} \nu_\beta^{-1}(s) + p(s) e^{-\gamma t},$$

is also fast decaying, and this concludes the proof. □

Corollary 6.3 (LTQO)**.** *Under the same hypotheses of lemma 6.2, $\mathcal{L}^{\overline{\Lambda}}$ satisfies LTQO for all Λ.*

Proof. Take $A \subset \Lambda$, and $s \geqslant 0$. Let ρ_1 and ρ_2 be two fixed point for $T_t^{A(s)}$. Then, by triangle inequality, we have that

$$|\operatorname{tr} O_A(\rho_1 - \rho_2)| \leqslant |\operatorname{tr} O_A(\rho_1 - \rho_\infty^s)| + |\operatorname{tr} O_A(\rho_\infty^s - \rho_2)| \leqslant 2\,|A|^\delta\,\Delta_0.$$

□

Lemma 6.4. *Under the same hypoteses of lemma 6.2, denote by ρ_∞ the unique fixed point of T_t. Then we have*

$$|\operatorname{tr} O_A(\rho_\infty - \rho_\infty^s)| \leqslant |A|^\delta\,\Delta_0(s). \tag{27}$$

Proof. By triangle inequality

$$\begin{aligned}|\operatorname{tr} O_A(\rho_\infty - \rho_\infty^s))| \leqslant \\ &\left|\operatorname{tr} O_A(\rho_\infty - T_t^{\overline{A(s)}}(\rho_\infty)\right| \\ +&\left|\operatorname{tr} O_A(T_t^{\overline{A(s)}}(\rho_\infty) - \rho_\infty^s)\right|.\end{aligned}$$

The first term can be bounded using lemma 5.5 and lemma 5.6:

$$\left|\operatorname{tr} O_A\big(\rho_\infty - T_t^{\overline{A(s)}}(\rho_\infty)\big)\right| = \left|\operatorname{tr} O_A\big(T_t(\rho_\infty) - T_t^{\overline{A(s)}}(\rho_\infty)\big)\right| \leqslant \|O_A\|\,|A|\,\frac{J}{v}e^{vt}\nu_\beta^{-1}(s).$$

The second one using the global rapid mixing hypotesis:

$$\left|\operatorname{tr} O_A\big(T_t^{\overline{A(s)}}(\rho_\infty) - \rho_\infty^s\big)\right| \leqslant |A|^\delta\,p(s)e^{-\gamma t}.$$

By making the same choice of $t = t(s)$ as in Lemma 6.2, we get the desired bound. □

Corollary 6.5 (Approximated frustation-freeness)**.** *Under the same hypoteses of lemma 6.2, denote by ρ_∞ the unique fixed point of T_t, and by ρ a periodic point of $T_t^{A(s)}$. Then we have*

$$|\operatorname{tr} O_A(\rho_\infty - \rho)| \leqslant 2\,|A|^\delta\,\Delta_0(s). \tag{28}$$

Proof. By triangle inequality and Lemma 6.2 and 6.4:

$$|\operatorname{tr} O_A(\rho_\infty - \rho))| \leqslant |\operatorname{tr} O_A(\rho_\infty - \rho_\infty^s)| + |\operatorname{tr} O_A(\rho_\infty^s - \rho)| \leqslant 2\,|A|^\delta\,\Delta_0(s).$$

□

6.2 Decay of correlations and mutual information

As a straightforward consequence of the results of the previous section, the hypotheses on $\mathcal{L}$ imply that its fixed points have a particular character: they verify a fast decay of correlations, meaning that the correlations between two spatially separated regions is fast-decaying in distance. The decay rate will be given by the decay rate of the interactions: it will be exponential if $\mathcal{L}$ verifies (LR-1), polynomial (or super-polynomial) if it verifies (LR-2).

There are a number of possibile measures of correlations between spatially separated regions. We shall present now three of them:

Definition 6.6. *Given a bipartite density matrix ρ_{AB}, denote by ρ_A (resp. ρ_B) the reduced density matrix on the subsystem A (resp. B). Then we define the following measures of correlations:*

- Covariance correlation*:*

$$\begin{aligned} C(A:B) &= \max_{\substack{M\in\mathcal{B}(H_A),N\in\mathcal{B}(H_B)\\ \|M\|\leqslant 1,\|N\|\leqslant 1}} \left|\mathrm{tr}\left[M\otimes N(\rho_{AB}-\rho_A\otimes\rho_B)\right]\right| \\ &= \max_{\substack{M\in\mathcal{B}(H_A),N\in\mathcal{B}(H_B)\\ \|M\|\leqslant 1,\|N\|\leqslant 1}} \left|\langle M\otimes N\rangle - \langle M\rangle\langle N\rangle\right|; \end{aligned}$$

  where $\langle O\rangle = \mathrm{tr}(O\rho_{AB})$ is the expectation value of the observable O acting on ρ_{AB}.

- Trace distance correlation*:*

$$\begin{aligned} T(A:B) &= \|\rho_{AB}-\rho_A\otimes\rho_B\|_1 \\ &= \max_{\substack{O\in\mathcal{B}(H_{AB})\\ \|O\|\leqslant 1}} \left|\mathrm{tr}\left[O(\rho_{AB}-\rho_A\otimes\rho_B)\right]\right|. \end{aligned}$$

- Mutual information correlation*:*

$$I(A:B) = S(\rho_A)+S(\rho_B)-S(\rho_{AB});$$

  where $S(\rho) = -\mathrm{tr}(\rho\ln\rho)$ is the von Neumann entropy entropy of the state ρ.

There are easy-to-show relationships between these correlations measures (see [43]):

$$C(A:B)\leqslant T(A:B)\leqslant 2\sqrt{I(A:B)}.$$

Moreover, by applying Fannes inequality [43, Box 11.2] we obtain that if $T(A:B)\leqslant\frac{1}{2e}$, then

$$I(A:B)\leqslant T(A:B)[\log D_{AB}-\log T(A:B)],$$

where D_{AB} is the dimension of the Hilbert space associated to the system AB.

These bounds imply that, if we can show that $T(A:B)$ is fast-decaying in the distance between regions A and B increase, then also the other two correlations measures $C(A:B)$ and $I(A:B)$ will be fast-decaying, as stated in theorem 4.7.

Proposition 6.7. *Under the same hypoteses of Lemma 6.2, denote by ρ_∞ the unique fixed point of T_t. Fix two regions A and $B \subset \Lambda$, call d_{AB} the distance between them, and suppose that $d_{AB} > 2R$, where R is the size of the boundary of $\mathcal{L}$. Then we have that*

$$T(A:B) \leqslant 3(|A| + |B|)^\delta \Delta_0 \left(\frac{d_{AB}}{2} - R \right),$$

where the correlations are calculated with respect to ρ_∞, and δ and Δ_0 are defined in Lemma 6.2.

Proof. The proposition is a simple consequence of Lemma 6.4. Call $C = A \cup B$, and denote by ρ_{AB} the reduced density matrix of ρ_∞ over C, and by ρ_A and ρ_B the reduced state on A and B, respectively.

Consider ρ^s_∞ the unique fix point of $T_t^{\overline{C(s)}}$. If $s \leqslant \frac{d_{AB}}{2} - R$, then $C(s)$ has two disjoint components corresponding to $A(s)$ and $B(s)$, and thus ρ^s_∞ decomposes as a tensor product of the form $\rho_\infty^{A(s)} \otimes \rho_\infty^{B(s)}$. Call ρ^s_A and ρ^s_B the reduced state of $\rho_\infty^{A(s)}$ and $\rho_\infty^{B(s)}$ over A and B, respectively.

By Lemma 6.4, we have that, for any observable O_C supported on C,

$$\left| \operatorname{tr} O_C (\rho_\infty - \rho_\infty^{A(s)} \otimes \rho_\infty^{B(s)}) \right| \leqslant |C|^\delta \Delta_0(s).$$

This implies that

$$\| \rho_{AB} - \rho^s_A \otimes \rho^s_B \|_1 \leqslant |C|^\delta \Delta_0(s).$$

Since the trace norm does not increases under the partial trace, then

$$\| \rho_A - \rho^s_A \|_1 \leqslant |C|^\delta \Delta_0(s),$$

and the same holds for B. This in turn implies that

$$\| \rho_A \otimes \rho_B - \rho^s_A \otimes \rho^s_B \|_1 \leqslant 2 |C|^\delta \Delta_0(s),$$

and by applying the triangle inequality we obtain the desired result. □

6.3 Step 2: from global to local rapid mixing

As a second step in the proof, we show that the hyphoteses on $\mathcal{L}$ we have made imply local rapid mixing.

Proposition 6.8 (From global to local rapid mixing)**.** *Let $\mathcal{L}$ be a uniform family of Linbdladians with unique fixed point. Then, if it satisfies condition (11), it also satisfies local rapid mixing.*

Proof. Let O_A be an observable supported on A with $\|O_A\| = 1$. Fix $s \geqslant 0$, and call $B = A(s)$. Then, by the triangle inequality, we can bound the norm of $(T_t^* - T_\infty^*)$ as follows

$$\begin{aligned} \|(T_t^* - T_\infty^*) O_A\| \leqslant \\ \left\| (T_t^* - T_t^{\overline{B}*}) O_A \right\| + \left\| (T_t^{\overline{B}*} - T_\infty^{\overline{B}*}) O_A \right\| + \left\| (T_\infty^{\overline{B}*} - T_\infty^*) O_A \right\|. \end{aligned} \tag{29}$$

We can bound the first and the fourth term in the sum using lemma 5.5 and lemma 5.6

$$\left\|(T_t^* - T_t^{\overline{B}*})O_A\right\| \leqslant |A| \frac{J}{v}(e^{vt} - 1 - vt)e^{-\beta s}. \tag{30}$$

The second term is simply bounded by the contraction of $T_t^{\overline{B}}$, and by the global rapid mixing hypotesis we have that

$$\left\|(T_t^{\overline{B}*} - T_\infty^{\overline{B}*})O_A\right\| \leqslant \eta(T_t^{\overline{B}}) \leqslant k(|A|)h(s)e^{-t\gamma}, \tag{31}$$

where $h(s)$ is bounded by a polynomial in s.

Finally, the third term is bounded by using Lemma 6.4:

$$\left\|(T_\infty^{\overline{B}*} - T_\infty^*)O_A\right\| = |\mathrm{tr}\, O_A(\rho_\infty^s - \rho_\infty)| \leqslant |A|^\delta \Delta_0(s). \tag{32}$$

By putting bounds (30), (31) and (32) into equation (29), we obtain the following bound

$$\eta^A(T_t) \leqslant \frac{J}{v}|A|(e^{vt} - 1 - vt)e^{-\beta s} + |A|^\delta \Delta_0(s) + k(|A|)h(s)e^{-t\gamma},$$

We want to show that, for each time $t \geqslant 0$, we can choose $s = s(t)$ in such a way that both $e^{vt}e^{-\beta s}$ and $e^{-t\gamma}h(s)$ are exponentially decaying in t, i.e. as $e^{-\varepsilon t}$ for some fixed $\varepsilon > 0$.

For each $t \geqslant 0$, let us choose an $s_0 = s_0(t)$ such that

$$(e^{vt} - 1 - vt)e^{-\beta s} \leqslant e^{-\varepsilon t} \quad \forall s \geqslant s_0.$$

We can choose $s_0(t)$ such that it grows linearly: $s_0(t) = t(v+\varepsilon)/\beta$. Let us choose a second $s_1 = s_1(t)$ such that

$$h(s)e^{-\gamma t} \leqslant e^{-\varepsilon t} \quad \forall s \leqslant s_1.$$

We can choose $s_1(t)$ such that $\log(s_1(t)) = O(t)$.

Since $s_0(t)$ grows at most linearly in s, while $s_1(t)$ grows at least exponentially in t, we can always find and $s = s(t)$ such that

$$s_0(t) \leqslant s(t) \leqslant s_1(t) \quad \text{and } \Delta_0(s(t)) \leqslant e^{-\varepsilon t}.$$

With this choice of $s(t)$, we obtain that

$$\eta^A(T_t) \leqslant q(|A|)e^{-\varepsilon t},$$

where $q(\cdot)$ grows polynomially. This completes the proof. □

6.4 Step 3: from local rapid mixing to stability

We now prove that local rapid mixing alone implies stability. This is the last step in the proof of theorem 4.3, as we already proved in the previous sections that the hypotheses of theorem 4.3 imply local rapid mixing. However, the following result also stands independently: if a system can be shown to satisfy local rapid mixing by other means, it will also be stable.

Theorem 6.9. *Let $\mathcal{L}$ be a local Lindbladian satisfying local rapid mixing, and having a unique fixed point ρ_∞ such that*

$$T_\phi = T_\infty = |\rho_\infty\rangle\!\rangle\langle\!\langle \mathbb{1}| \,.$$

Then, using the same notation as in theorem 4.3, for all observables O_A supported on $A \subset \Lambda$ we have that

$$\|O_0(t) - O_1(t)\| \leqslant c(|A|)\, \|O_A\|\, \varepsilon, \tag{33}$$

for some c not depending on the system size, independent of t, and polynomial in $|A|$.

Proof. Recall that $O_1 = T_t^*(O_A)$ and $O_2 = S_t^*(O_A)$. Let us write the difference $O_0 - O_1$ using the integral representation

$$O_0(t) - O_1(t) = \int_0^t S_{t-s}^* E^* T_s^*(O_A)\, \mathrm{d}s = \sum_u \sum_r \int_0^t S_{t-s}^* E_u(r)^* T_s^*(O_A)\, \mathrm{d}s,$$

which implies

$$\|O_0(t) - O_1(t)\| \leqslant \sum_u \sum_r \int_0^t \|E_u(r)^* O_0(s)\|\, \mathrm{d}s,$$

where we used the fact that S_t is a contraction.

Fix a site u and a positive r, and call $\delta = \mathrm{dist}(A, b_u(r))$. We can split the integral at a time t_0 (to be fixed later, depending on δ). We bound the first part of the integral with Lieb-Robinson bounds:

$$\int_0^{t_0} \|E_u(r)^* O_0(s)\|\, \mathrm{d}s \leqslant \varepsilon e(r)\, \|O_A\|\, |A|\, \frac{e^{v t_0} - v t_0 - 1}{v \nu_\mu(\delta)}.$$

Now pick $t_0 = t_0(\delta)$ such that

$$\nu_\mu^{-1}(\delta) \frac{e^{v t_0} - v t_0 - 1}{v} \leqslant \nu_{\mu/2}^{-1}(\delta).$$

We can choose $t_0(\delta) = \frac{\mu}{2} \frac{\log v}{v} \delta = O(\delta)$.

If $t \leqslant t_0(\delta)$, then we have bounded the entire integral, and we are done. Otherwise, we treat the second part of the integral as follows:

$$\begin{aligned}
\int_{t_0(\delta)}^t \|E_u(r)^* O_0(s)\|\, \mathrm{d}s &= \int_{t_0(\delta)}^t \|E_u(r)^* (O_0(s) - T_\infty^*(O_A))\|\, \mathrm{d}s \\
&\leqslant \varepsilon e(r)\, \|O_A\| \int_{t_0(\delta)}^\infty \eta^A(T_s)\, \mathrm{d}s \\
&\leqslant \varepsilon e(r)\, \|O_A\|\, q(|A|) \int_{t_0(\delta)}^\infty e^{-\gamma s}\, \mathrm{d}s \\
&= \varepsilon e(r)\, \|O_A\|\, q(|A|) \frac{1}{\gamma} e^{-\gamma t_0(\delta)}
\end{aligned}$$

where we used

$$E_u^*(r)T_\infty^*(O_A) = E_u^*(r)\,|\mathbb{1}\rangle\!\rangle\langle\!\langle\rho_\infty|\,|O_A\rangle\!\rangle = \langle\!\langle\rho_\infty|O_A\rangle\!\rangle\, E_u^*(r)\,|\mathbb{1}\rangle\!\rangle = 0$$

together with the local rapid mixing hypothesis.

Since $t_0(\delta)$ is linear δ, we have that

$$\widetilde{g}(\delta) = e^{-\frac{\mu\delta}{2}} + \frac{1}{\gamma}e^{-\gamma t_0(\delta)}$$

is exponentially decaying.

Putting the integral back together, we obtain

$$\int_0^t \|E_u(r)^* O_0(s)\|\,\mathrm{d}s \leqslant \varepsilon\,\|O_A\|\,q_1(|A|)e(r)\widetilde{g}(\delta).$$

Returning to the sum, we have proved that

$$\|O_0(t) - O_1(t)\| \leqslant q_1(|A|)\varepsilon\,\|O_A\| \sum_u \sum_r e(r)\widetilde{g}(\mathrm{dist}(A, b_r(u))).$$

It suffices to show that the sum on the r.h.s. is finite (and independent of system size). Let us decompose the sum as follows

$$\begin{aligned}
&\sum_u \sum_r e(r)\widetilde{g}(\mathrm{dist}(A, b_r(u))) \\
&\quad = \sum_{\mathrm{dist}(u,A)=0} \sum_r e(r)\widetilde{g}(0) + \sum_{d>0} \sum_{\mathrm{dist}(u,A)=d} \left(\sum_{r=0}^{d} e(r)\widetilde{g}(d-r) + \sum_{r=d+1}^{\infty} e(r)\widetilde{g}(0) \right) \\
&\quad = \widetilde{g}(0)\,|A| \sum_r e(r) + \sum_{d>0} q(d) \left(\sum_{r=0}^{d} e(r)\widetilde{g}(r-d) + \widetilde{g}(0) \sum_{r=d+1}^{\infty} e(r) \right),
\end{aligned}$$

where $q(d) = |\{u : \mathrm{dist}(u, A) = d\}|$ grows polynomially in d.

The first term is clearly bounded, since $e(r)$ is summable.

Since e and $\widetilde{g}$ are both exponentially decaying functions, their discrete convolution $e \star \widetilde{g}(d) = \sum_{r=0}^{d} e(r)\widetilde{g}(r-d)$ is also exponentially decaying, and consequently summable against any polynomial. The same holds for $\sum_{r>d} e(r)$. This proves that the second term is also bounded.

Calling

$$c(|A|) = q_1(|A|)\left[\widetilde{g}(0)\,|A| \sum_r e(r) + \sum_{d>0} q(d)\left(e \star \widetilde{g}(d) + \widetilde{g}(0) \sum_{r>d} e(r)\right)\right],$$

we can finally derive the claimed bound. □

6.5 Power-law decay

As we stated before, the results and proofs presented still hold, with the obvious modifications, when $\mathcal{L}$ has quasi-local or power-law interactions. In the latter case, this is only true when some compatibility condition between different properties of $\mathcal{L}$ is verified. We have avoided to get into such generalization before, in order not to overload the exposition: is it time now to present them.

Definition 6.10 (Compatibility condition)**.** *Let $\mathcal{L}$ be a local Lindbladian having power-law decay with rate α. Suppose $\mathcal{L}$ satisfies* (LR-2) *and global rapid mixing GRM(γ, δ). Let μ and v be the Lieb-Robinson constants for $\mathcal{L}$ defined in Assumption 5.1 and β the constant defined in Lemma 5.5. Then we say that $\mathcal{L}$ satisfies the compatibility condition if the following conditions are satisfied:*

$$\alpha > 3D + 2, \tag{CC-1}$$

$$\beta > \frac{v}{\gamma}\left(v + \gamma + D\delta\right), \tag{CC-2}$$

$$\beta \geqslant v + \gamma - D\delta. \tag{CC-3}$$

Moreover, the if E defined in Theorem 4.3 is decaying polynomially and not exponentially, with a decay rate of η, it must verify $\eta > \mu$ for the theorem to hold.

Observation 6.11. If $\mathcal{L}$ has quasi-local interactions, than α is arbitrarily large. Consequently, since β is linear in α, we have that in this case $\mathcal{L}$ always satisfies the compatibility condition.

Under the compatibility condition, all the results presented until now still hold true. We will now show this in the cases in which we made explicit use of condition (LR-1), and give the needed modifications to the proofs of Lemma 6.2, Proposition 6.8 and Theorem 6.9 in order to make them work also in the case of power-law decay.

From now on, we will always assume that $\mathcal{L}$ verifies (LR-2) and the compatibility condition.

Proof of Lemma 6.2. Using the same notation of the original proof, we have that for all $t \geqslant 0$:

$$|\operatorname{tr} O_A(\rho - \rho_\infty^s)| \leqslant \|O_A\| \left(|A|\, J \frac{e^{vt}}{v\nu_\beta(s)} + |A(s)|^\delta\, e^{-\gamma t} \right).$$

Let us fix a positive k to be determined later, and consider the following scaling of t with s:

$$t = t(s) = k \log(1 + s);$$

in such a way that

$$\frac{e^{vt}}{\nu_\beta(s)} = \nu_{vk-\beta}(s) \quad \text{and} \quad e^{\gamma t} = \nu_{\gamma k}(s).$$

We want to show that, with this choice of $t = t(s)$, the function

$$\Delta_0(s) = \frac{1}{|A|^{1\vee\delta}} \left(|A|\, J \frac{e^{vt}}{v\nu_\beta(s)} + |A(s)|^\delta\, e^{-\gamma t} \right)$$

is fast decaying.

The first term in the sum is bounded as follows:

$$|A|\, J \frac{e^{vt}}{v\nu_\beta(s)} \leqslant |A|\, J \nu_{vk-\beta}(s);$$

the second one, is bounded by

$$|A(s)|^\delta\, e^{-\gamma t} \leqslant |A|^\delta\, \nu_{D\delta-\gamma k}(s).$$

Both terms are decaying if and only if there exist a choice of k such that $vk - \beta$ and $D\delta - \gamma k$ are simultaneously negative. In this case, the decay rate is given by the slowest of the two, and the best rate attainable is the following:

$$\delta_0 = \sup_{k>0} \min(\beta - vk, \gamma k - D\delta).$$

The optimal choice for k is then given by $k = \bar{k}$, where

$$\bar{k} = \frac{\beta + D\delta}{v + \gamma},$$

in such a way that

$$\delta_0 = \beta - v\bar{k} = \gamma\bar{k} - D\delta = \frac{\gamma}{v+\gamma}\beta - \frac{v}{v+\gamma}D\delta.$$

Such δ_0 is bounded away from zero if and only if

$$\beta > \frac{v}{\gamma} D\delta, \tag{34}$$

which is true because of (CC-2). □

Proof of Proposition 6.8. With the same notation of the original proof, we can prove that for each $s \geqslant 0$:

$$\eta^A(T_t) \leqslant \frac{J}{v} |A| (e^{vt} - 1 - vt)\nu_\beta^{-1}(s) + |A|^\delta \Delta_0(s) + k(|A|)\nu_{D\delta}(s)e^{-\gamma t}.$$

We want to show that, for each time $t \geqslant 0$, we can choose $s = s(t)$ in such a way that the r.h.s. is exponentially decaying in t.

Fix a positive k (to be determined later), and consider

$$s(t) = e^{kt} - 1,$$

in such a way that

$$e^{vt}\nu_\beta^{-1} = e^{(v-\beta k)t}; \quad \Delta_0(s) = e^{-\delta_0 kt}; \quad e^{-\gamma t}\nu_{D\delta} = e^{(kD\delta - \gamma)t};$$

where δ_0 was defined in the previous proof. We want to show that we can choose k in such a way that all the exponential appearing above are decaying, i.e. their exponent are negative. The optimal decay rate is then given by

$$\widetilde{\gamma} = \sup_{k>0} \min\left(\beta k - v, \gamma - kD\delta, \delta_0 k\right).$$

We can choose, as in the previous proof,

$$k = \bar{k} = \frac{\beta + D\delta}{v + \gamma},$$

such that, since (CC-3) implies that $\bar{k} \geqslant 1$.

$$\widetilde{\gamma} = \delta_0 \min(1, \bar{k}) = \delta_0.$$

This shows that, with this choice of k, $\widetilde{\gamma}$ is always positive and scales at least linearly as α increases. □

Proof of Theorem 6.9. Following the same steps as in the original proof, we need to choose $t_0(\delta) = k\log(1+\delta) = O(\log(\delta))$ for some positive k (to be determined later), in order to bound $\widetilde{g}$.

With this choice, we have that

$$e^{vt_0}\nu_\mu^{-1}(\delta) = \nu_{vk-\mu}(\delta) \quad \text{and} \quad e^{-\widetilde{\gamma}t_0} = \nu_{-\widetilde{\gamma}k},$$

where $\widetilde{\gamma}$ is defined in the previous proof.

Then we have that $\widetilde{g}$ has a maximum decay rate of

$$\widetilde{\varepsilon} = \sup_{k\geqslant 0} \min(\mu - vk, \widetilde{\gamma}k).$$

The optimal choice of k is $k = \widetilde{k} = \frac{\mu}{\widetilde{\gamma}-v}$, in such a way that

$$\widetilde{\varepsilon} = \frac{\widetilde{\gamma}\mu}{\widetilde{\gamma}-v} > \mu.$$

$\widetilde{k}$ is positive whenever $\widetilde{\gamma} > v$. Recall that $\widetilde{\gamma}$ is chosen to be equal to $\delta_0 \min(1, \bar{k}) = \delta_0$ (because condition (CC-3) implies $\bar{k} \geqslant 1$).

Then $\widetilde{\gamma} > v$ if and only if

$$\beta > \frac{v}{\gamma}\left(v + \gamma + D\delta\right),$$

which is exactly condition (CC-2).

Then, we need to show that

$$\sum_u \sum_r e(r)\widetilde{g}(\mathrm{dist}(A, b_r(u)))$$

is finite and independent of system size. Suppose that $e(r)$ decays as $(1+r)^{-\eta}$. Then $e \star \widetilde{g}(d)$ has a polynomial decay of exponent $\min(\eta, \widetilde{\varepsilon}) > \mu$ (since we have assumed $\eta > \mu$).

Since $q(d)$ grows as $(1+d)^{D-1}$, for $q \cdot (e \star \widetilde{g})$ to be summable it is sufficient that $\mu > D$. Similarly, for $\sum_{r>d} e(r)$ to be summable against $q(d)$, it suffices that $\eta > \mu > D+1$. Such a choice for μ is possible whenever $\alpha > 3D+2$, i.e. when the system satisfies (CC-1).

The rest of the proof is identical to the one for exponentially decaying interactions. □

7 Glauber dynamics

7.1 Quantum embedding of Glauber dynamics

As an example of a non-trivial dynamics for which we can now prove stability using our results, we turn to one of the most studied dynamics in classical statistical mechanics: Glauber dynamics, a Markov process that samples thermal states of local (classical) Hamiltonians on lattices. Apart from being an interesting model in itself, it has important applications in Monte-Carlo Markov chain algorithms for numerical many-body physics [34]. Determining whether Glauber dynamics is stable against noise or errors is therefore an important – and, as far as we are aware, still open – question.

In this section, we present a natural embedding of Glauber dynamics into the Linbdlabian setting, showing how this embedded dynamics inherits properties from the classical Markov chain. We will then apply the results of section 4 to prove, in the appropriate regime, stability of Glauber dynamics.

We will consider a lattice spin system over $\Gamma = \mathbb{Z}^D$ or $\Gamma = (\mathbb{Z}/L\mathbb{Z})^D$, with (classical) configuration space of a single spin a finite set S. For simplicity, we will consider the case $S = \{+1, -1\}$. For each $\Lambda \subset \Gamma$, we will denote by Ω_Λ the space of configurations over Λ, namely S^Λ. Λ^c will denote the complementary of Λ in Γ, namely $\Gamma \setminus \Lambda$.

Definition 7.1. *A finite range, translation-invariant potential $\{J_A\}_{A\subset\Gamma}$ is a family of real functions indexed by the non empty finite subsets of Γ verifying the following properties:*

1. *$J_A : \Omega_A \to \mathbb{R}$.*
2. *For all $A \subset \Gamma$ and all $x \in \Gamma$:*

$$J_A(\sigma) = J_{A+x}(\eta) \quad \text{if} \quad \sigma(y+x) = \eta(y) \quad \forall y \in A.$$

3. *There exists a positive $r > 0$ such that $J_A = 0$ if $\operatorname{diam} A > r$, called range of interaction.*

Given a finite range, translation-invariant potential, we can define a Hamiltonian for each finite lattice $\Lambda \subset \Gamma$ and each boundary condition *$\tau \in \Omega_{\Lambda^c}$ by*

$$H_\Lambda^\tau(\sigma) = -\sum_{A\cap\Lambda\neq 0} J_A(\sigma \times \tau) \quad \forall \sigma \in \Omega_\Lambda,$$

where $\sigma \times \tau$ is the configuration that agrees with σ over Λ and with τ over Λ^c. For each such Hamiltonian, we define the Gibbs state state as

$$\mu_\Lambda^\tau(\sigma) = (Z_\Lambda^\tau)^{-1} \exp(-H_\Lambda^\tau(\sigma)),$$

where Z_Λ^τ is a normalizing constant.[10] *The convex hull of the set of Gibbs states over Λ will be denoted by $\mathcal{G}(\Lambda)$:*

$$\mathcal{G}(\Lambda) = \operatorname{conv}\{\mu_\Lambda^\tau \,|\, \tau \in \Omega_{\Lambda^c}\}.$$

Definition 7.2. *The Glauber dynamics for a potential J is the Markov process on Ω_Λ with the following generator:*

$$(Q_\Lambda f)(\sigma) = \sum_{x\in\Lambda} c_J(x,\sigma)\nabla_x f(\sigma),$$

where $\nabla_x f(\sigma)$ if defined as $f(\sigma^x) - f(\sigma)$, and σ^x is the configuration obtained by flipping the spin at position x:

$$\sigma^x(y) = \begin{cases} \sigma(y) & \text{if } x \neq y \\ -\sigma(x) & \text{if } x = y. \end{cases}$$

The numbers $c_J(x,\sigma)$ are called transition rates and must verify the following assumptions:

[10] Following [38], in our notation we have incorporated the usual inverse temperature parameter β directly into the potential J.

1. *Positivity and boundedness: There exist positive constants c_m and c_M such that:*
$$0 < c_m \leqslant c_J(x,\sigma) \leqslant c_M < \infty \quad \forall x, \sigma.$$
2. *Finite range: $c_J(x,\cdot)$ depends only on spin values in $b_r(x)$.*
3. *Translation invariance: for all $k \in \Gamma$,*
$$c_J(x,\sigma') = c_J(x+k,\sigma) \quad \text{if} \quad \sigma'(y) = \sigma(y+k) \quad \forall y.$$
4. *Detailed balance: for all $x \in \Gamma$ and all σ*
$$\exp\left(-\sum_{A \ni x} J_A(\sigma)\right) c_J(x,\sigma) = c_J(x,\sigma^x) \exp\left(-\sum_{A \ni x} J_A(\sigma^x)\right).$$

These assumptions are sufficient to ensure that Q_Λ generates a Markov process which has the Gibbs states over Λ as stationary points.

Definition 7.3. *A quantum embedding of the classical Glauber dynamics for a potential J is generated by the following Lindblad operators*

$$L_{x,\eta} = \sqrt{c_J(x,\eta)}\, |\eta^x\rangle\langle\eta| \otimes \mathbb{1}, \quad \forall x \in \Lambda, \forall \eta \in \Omega_{b_x(r)}; \tag{35}$$
$$\mathcal{L}_{x,\eta}(\rho) = L_{x,\eta}\rho L_{x,\eta}^* - \frac{1}{2}\left\{\rho, c_J(x,\eta)\,|\eta\rangle\langle\eta|\right\};$$
$$\mathcal{L}_\Lambda(\rho) = \sum_{x\in\Lambda}\sum_{\eta} L_{x,\eta}\rho L_{x,\eta}^* - \frac{1}{2}\{\rho, K\}, \quad K = \sum_\sigma \left(\sum_x c_J(x,\sigma)\right)|\sigma\rangle\langle\sigma|; \tag{36}$$

plus a dephasing channel acting independently and uniformly on all sites $x \in \Lambda$:

$$D_{x,0} = \sqrt{\gamma}\,|0\rangle\langle 0|, \quad D_{x,1} = \sqrt{\gamma}\,|1\rangle\langle 1|, \quad \mathcal{D}(\rho) = \sum_{x\in\Lambda}\sum_{i=0,1} D_{x,i}\rho D_{x,i}^* - |\Lambda|\,\gamma\rho. \tag{37}$$

$\mathcal{L}_\Lambda$ verifies translational invariance because the transition rates c_J do, and it easy to see that this family of Lindbladians is uniform.

Observation 7.4. Take $|\alpha\rangle\langle\beta|$ an element of the computational basis, and call $d(\alpha,\beta)$ the Hamming distance between α and β. Then it holds that

$$\mathcal{D}(|\alpha\rangle\langle\beta|) = -\gamma d(\alpha,\beta)\,|\alpha\rangle\langle\beta|.$$

In other words, $\mathcal{D}$ is a Schur multiplier in the computational basis, with Schur matrix given by $(-\gamma d(\alpha,\beta))_{\alpha,\beta}$.

On the other hand, we have that for all x:

$$\sum_{\eta\in\Omega_{b_x(r)}} \mathcal{L}_{x,\eta}(|\alpha\rangle\langle\beta|) = \begin{cases} c_J(x,\alpha)\left(|\alpha^x\rangle\langle\beta^x| - |\alpha\rangle\langle\beta|\right) & \text{if } \alpha|_{b_x(r)} = \beta|_{b_x(r)}, \\ -\frac{1}{2}\left(c_J(x,\alpha) + c_J(x,\beta)\right)|\alpha\rangle\langle\beta| & \text{otherwise.} \end{cases} \tag{38}$$

Since $d(\alpha^x,\beta^x) = d(\alpha,\beta)$, $[\mathcal{D}, \sum_\eta \mathcal{L}_{x,\eta}] = 0$ for all $x \in \Lambda$, and in particular $\mathcal{D}$ and $\mathcal{L}_\Lambda$ commute.

This quantum dissipative system inherits various properties from its classical counterpart.

Definition 7.5. *Let μ be a full-rank positive state. Call*

$$\Gamma_\mu(\rho) = \mu^{\frac{1}{2}} \rho \mu^{\frac{1}{2}}.$$

We say that $\mathcal{L}$ is in detailed balance [48] with respect to μ if $\Gamma_\mu \circ \mathcal{L} = \mathcal{L}^* \circ \Gamma_\mu$.

Proposition 7.6. *Let μ_Λ^τ be a Gibbs state over Λ. Then $\mathcal{L}_\Lambda$ and $\mathcal{D}$ are in detailed balance with respect to μ_Λ^τ.*

Proof. Note that $\Gamma_{\mu_\Lambda^\tau}$ is a Schur multiplier in the computational basis:

$$\Gamma_{\mu_\Lambda^\tau}(|\eta_1\rangle\langle\eta_2|) = \mu_\Lambda^\tau(\eta_1)^{\frac{1}{2}} \mu_\Lambda^\tau(\eta_2)^{\frac{1}{2}} |\eta_1\rangle\langle\eta_2|.$$

From the detailed balance condition for the transition rates $c_J(x,\sigma)$, it follows that for all $x \in \Lambda$, denoting $\mathcal{L}_x = \sum_{\eta \in \Omega_{b_x(r)}} \mathcal{L}_{x,\eta}$,

$$\begin{aligned}
&\Gamma_{\mu_\Lambda^\tau} \circ \mathcal{L}_x \circ \Gamma^{-1}_{\mu_\Lambda^\tau}(|\eta_1\rangle\langle\eta_2|) \\
&\quad = \delta^x_{\eta_1,\eta_2} \left(c_J(x,\eta_1) \frac{\mu_\Lambda^\tau(\eta_1^x)}{\mu_\Lambda^\tau(\eta_1)} \right) |\eta_1^x\rangle\langle\eta_2^x| - \frac{c_J(x,\eta_1) + c_J(x,\eta_2)}{2} |\eta_1\rangle\langle\eta_2| \\
&\quad = \delta^x_{\eta_1,\eta_2} c_J(x,\eta_1^x) |\eta_1^x\rangle\langle\eta_2^x| - \frac{c_J(x,\eta_1) + c_J(x,\eta_2)}{2} |\eta_1\rangle\langle\eta_2| \\
&\quad = \mathcal{L}_x^*(|\eta_1\rangle\langle\eta_2|)),
\end{aligned}$$

where

$$\delta^x_{\eta_1,\eta_2} = \begin{cases} 1 & \text{if } \eta_1|_{b_x(r)} = \eta_2|_{b_x(r)} \\ 0 & \text{otherwise.} \end{cases}$$

To prove detailed balance for $\mathcal{D}$, note that Schur multipliers commute, thus $[\mathcal{D}, \Gamma_\mu] = 0$. This, together with the fact that $\mathcal{D}^* = \mathcal{D}$, implies that $\mathcal{D}$ is in detailed balance w.r.t. μ_Λ^τ. □

The above proposition implies that Gibbs states are stationary states for the quantum Glauber dynamics. Let us prove that there are no other fixed points apart from the classical ones (i.e. states that are diagonal in the computational basis). Clearly, $\mathcal{D}$ has all classical states as stationary points. We just have to check $\mathcal{L}_\Lambda$.

Proposition 7.7. *The set of fixed points of $\mathcal{L}_\Lambda$ is equal to $\mathcal{G}(\Lambda)$, the set of Gibbs states over Λ.*

Proof. Let ρ be a fixed point of $\mathcal{L}_\Lambda$. We want to prove that ρ is diagonal, i.e. that it is of the form

$$\rho = \sum_\sigma p_\sigma |\sigma\rangle\langle\sigma|.$$

Consider a non-diagonal element $|\alpha\rangle\langle\beta|$, and suppose $\alpha(x) \neq \beta(x)$ for some $x \in \Lambda$. Then, from equation (38), we have that for all $y \in b_x(r)$,

$$\mathcal{L}_y(|\alpha\rangle\langle\beta|) = -\frac{1}{2}(c_J(y,\alpha) + c_J(y,\beta)) |\alpha\rangle\langle\beta|.$$

For $y \notin b_x(r)$, $\mathcal{L}_y$ is not supported on x, and thus cannot change the configuration there. This implies that the evolution cannot change the configurations over

the set $\Delta(r)$, where $\Delta = \{x \in \Lambda \,|\, \alpha(x) \neq \beta(x)\}$. In turn, this implies that $\mathcal{L}_\Delta$ commutes with $\mathcal{L} - \mathcal{L}_\Delta$ (since it acts as a Schur multiplier whose entries depend only on the sites in $\Delta(r)$). Finally, this means that

$$\begin{aligned}\left\|e^{t\mathcal{L}_\Lambda}(|\alpha\rangle\langle\beta|)\right\|_1 &\leqslant \left\|e^{t\mathcal{L}_\Delta}(|\alpha\rangle\langle\beta|)\right\|_1 = \exp\left(-t\frac{1}{2}\left(\sum_{x\in\Delta} c_J(x,\alpha) + c_J(x,\beta)\right)\right)\\ &\leqslant \exp\left(-t\frac{1}{2}c_m d(\alpha,\beta)\right) \to 0.\end{aligned}$$

Since the off-diagonal elements are killed, ρ must be of the form $\sum_\sigma p_\sigma |\sigma\rangle\langle\sigma|$. Writing the equation $\mathcal{L}_\Lambda(\rho) = 0$ we obtain

$$\sum_\sigma \sum_x c_J(x,\sigma) p_\sigma |\sigma^x\rangle\langle\sigma^x| - \sum_\sigma \sum_x c_J(x,\sigma) p_\sigma |\sigma\rangle\langle\sigma| = 0,$$

which implies

$$\sum_x c_J(x,\sigma^x) p_{\sigma^x} = \sum_x p_\sigma c_J(x,\sigma).$$

The last equation is simply a rewriting of the fact that (p_σ) is a stationary distribution for Q_Λ, that is, it is exactly a Gibbs state on Λ. □

Since $\mathcal{L}_\Lambda$ and $\mathcal{L}_\Lambda + \mathcal{D}$ have the same stationary distributions, even locally, all properties that depend just on the structure of the fixed points sets will be shared by both: this is the case, for example, of frustration freeness (which we will prove next) and LTQO (which will be proved later).

Proposition 7.8. *$\mathcal{L}_\Lambda$ (and consequently $\mathcal{L}_\Lambda + \mathcal{D}$) is frustration free.*

Proof. By the previous proposition, we have that $\mathcal{X}_{\mathcal{L}_\Lambda} = \mathcal{G}(\Lambda)$. We know [34] that for Gibbs states it holds that

$$\Delta \subset \Lambda \Rightarrow \mathcal{G}(\Lambda) \subset \mathcal{G}(\Delta),$$

but this is exactly the frustration-freeness condition for $\mathcal{L}_\Lambda$. □

7.2 Stability of Glauber dynamics

We want to show that the contraction of the semigroup generated by $\mathcal{L}_\Lambda + \mathcal{D}$ can be controlled by the contraction of the classical Glauber dynamics. To fix notation, call $\mathcal{C} : \mathcal{A}_\Lambda \to \mathcal{A}_\Lambda$ the projector on the computational basis diagonal. $\mathcal{C}$ is a completely positive, trace preserving map, and it also verifies $\mathcal{C} = \lim_{t\to\infty} \exp(t\mathcal{D})$. Since $\mathcal{L}_\Lambda$ commutes with $\mathcal{D}$, it also commutes with $\mathcal{C}$. Then we can prove the following

Lemma 7.9. *If $T_t = \exp\left(t(\mathcal{L}_\Lambda + \mathcal{D})\right)$, then*

$$\eta(T_t) \leqslant \eta(T_t \circ \mathcal{C}) + \eta(\exp(t\mathcal{D})). \tag{39}$$

Proof. Fix an initial state ρ. Then we can write

$$\begin{aligned}\|T_t(\rho) - T_\infty(\rho)\|_1 &\leqslant \|T_t \circ \mathcal{C}(\rho) - T_\infty(\rho)\|_1 + \|T_t \circ (1-\mathcal{C})(\rho)\|_1\\ &\leqslant \|T_t \circ \mathcal{C}(\rho) - T_\infty \circ \mathcal{C}(\rho)\|_1 + \|\exp(t\mathcal{D}) \circ (1-\mathcal{C})(\rho)\|_1\\ &\leqslant \eta(T_t \circ \mathcal{C}) + \eta(\exp(t\mathcal{D})),\end{aligned}$$

where we have used the fact that $\mathcal{L}_\Lambda$ and $\mathcal{D}$ commute, and that the fixed points of $\mathcal{L}_\Lambda$ are invariant under $\mathcal{C}$. □

We know, because of theorem 4.2, that

$$\eta(\exp(t\mathcal{D})) \leqslant |\Lambda| e^{-\frac{\gamma}{2}t}, \tag{40}$$

and this implies the following result.

Corollary 7.10. *If the classical Glauber dynamics verifies global rapid mixing, then also the quantum embedded Glauber dynamics generated by $\mathcal{L}_\Lambda + \mathcal{D}$ does.*

Observation 7.11. Convergence rates of classical Glauber dynamics are a well studied subject. It is known that, in some regimes, classical Glauber dynamics verifies a Log Sobolev inequality with system-size independent Log Sobolev constant (for a review on the subject see [38]). In such situations the classical chain has a logarithmic mixing time, and thus verifies global rapid mixing.

For this class of classical dynamical systems it is possible to apply our main result 4.3. In particular, we can arbitrary perturb the transition rates $c_J(x,\sigma)$ by some $e(x,\sigma)$, not necessary preserving detailed balance. If we call $\mathcal{E}$ the maximum of $|e(x,\sigma)|$, the difference between the perturbed and the original evolution of local observables can be bounded by $\mathcal{E}$ times a factor depending on the size of the support of the observables taken into account.

Theorem 7.12. *Let Q_Λ the generator of a classical Glauber dynamics, having a unique fix point and verifying a Log Sobolev inequality with constant independent of system size. Let E be the generator of another classical Markov process of the form*

$$(Ef)(\sigma) = \sum_{x\in\Lambda} e(x,\sigma)\nabla_x f(\sigma).$$

Suppose that $\mathcal{E} = \sup_{x,\sigma} |e(x,\sigma)| < \infty$ and that $e(x,\cdot)$ has bounded support uniformly in x. Denote by T_t the evolution generated by Q_Λ and by S_t the evolution generated by $Q_\Lambda + E$. Then, for each function f supported on $A \subset \Lambda$, it holds that

$$\|T_t(f) - S_t(f)\|_\infty \leqslant c(|A|)\,\|f\|_\infty\,\mathcal{E},$$

for some $c(\cdot)$ independent of system size and polynomially growing.

Observation 7.13. It is known [36, 37] that the Ising model on $\mathbb{Z}^2$ or $(\mathbb{Z}/n\mathbb{Z})^2$ has a system size independent Log Sobolev constant for high temperatures (when the inverse temperature β is lower than the critical value β_c), or at any temperature in presence of an external magnetic field. In this regime the Glauber dynamics sampling the Ising model is stable (in the sense of theorem 4.3).

7.3 Weak mixing and LTQO

As a nice observation, though not necessary to prove Theorem 7.12, we want show that *weak mixing*, a condition on Gibbs states defined in [38], is equivalent to the LTQO condition given in section 6. The weak mixing conditions for 2 dimensional systems has been shown [37] to imply L_2 convergence of the corresponding Glauber dynamics.

Definition 7.14. *We say that the Gibbs measures in $\mathcal{G}(\Lambda)$ satisfy the* weak mixing *condition in $V \subset \Lambda$ if there exist constants C and m such that, for every*

subset $\Delta \subset V$, the following holds:

$$\sup_{\tau,\tau' \in \Omega_{V^c}} \left\| \mu^{\tau}_{V,\Delta} - \mu^{\tau'}_{V,\Delta} \right\|_1 \leqslant C \sum_{\substack{x \in \Delta, \\ y \in \partial^+_r V}} e^{-m \operatorname{dist}(x,y)}, \tag{41}$$

where $\partial^+_r V = \{x \in V^c \mid \operatorname{dist}(x, V) \leqslant r\}$ and $\mu^{\tau}_{V,\Delta} = \operatorname{tr}_{V \setminus \Delta} \mu^{\tau}_V$.

Proposition 7.15. *If $\mathcal{G}(\Lambda)$ verifies the weak mixing condition for each $V \subset \Lambda$, then $\mathcal{L}_\Lambda$ (and consequently $\mathcal{L}_\Lambda + \mathcal{D}$) verifies LTQO.*

Proof. Take $A \subset \Lambda$, $\ell \geqslant 0$, and call $V = A(\ell)$. The weak mixing condition for V implies that there exist constants C and m such that

$$\sup_{\tau,\tau' \in \Omega_{V^c}} \left\| \mu^{\tau}_{V,A} - \mu^{\tau'}_{V,A} \right\|_1 \leqslant C \sum_{\substack{x \in A, \\ y \in \partial^+_r V}} e^{-m \operatorname{dist}(x,y)} \leqslant C e^{-m\ell} |A| \, |\partial^+_r A(\ell)| \, .$$

This is the LTQO condition with $\Delta_0(\ell) = C e^{-m\ell} |A| \, |\partial^+_r A(\ell)|$. The bound, proven for states of the form μ^{τ}_V, can be extended by convexity to all $\mathcal{G}(V)$. Let $\eta_0, \eta_1 \in \mathcal{G}(V)$. By definition, η_0 and η_1 are convex combination of states of the form μ^{τ}_V, thus we can write

$$\eta_0 = \sum_i p_i \mu^{\tau_i}_V, \quad \eta_1 = \sum_j q_j \mu^{\sigma_j}_V, \quad \sum_i p_i = \sum_j q_j = 1; \quad p_i, q_j \geqslant 0.$$

Then we have

$$\begin{aligned} \|\eta_{0,A} - \eta_{1,A}\|_1 &= \left\| \sum_i p_i \mu^{\tau_i}_{V,A} - \sum_j q_j \mu^{\sigma_j}_{V,A} \right\|_1 \\ &= \left\| \sum_i p_i \Big(\sum_j q_j \mu^{\tau_i}_{V,A} \Big) - \sum_j q_j \Big(\sum_i p_i \mu^{\sigma_j}_{V,A} \Big) \right\|_1 \leqslant \sum_{i,j} p_i q_j \left\| \mu^{\tau_i}_{V,A} - \mu^{\sigma_j}_{V,A} \right\|_1 \\ &\leqslant \sup_{\tau,\sigma} \left\| \mu^{\tau}_{V,A} - \mu^{\sigma}_{V,A} \right\|_1 . \end{aligned}$$

□

8 Conclusions and open questions

We have shown stability of the evolution of local observables under local perturbations for Lindblad evolutions with unique fixed points under the assumption that the mixing time of the system scales logarithmically with the system size.

We have shown that fast-converging in such systems implies LTQO and a relaxed version of frustation freeness, together with exponential decay of correlations in their fixed points. It should be emphasized that Log Sobolev inequalities provide the exact type of convergence time estimates needed for stability.

The most important open question is what happens outside the rapid mixing regime, for example for systems for which the local convergence time depends on the global system size. This seems to be the case of state engineering of degenerate topological ordered states, such as topologically protected quantum codes.

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Appendix A The non-stable example

The following example will verify all hypotheses of theorem 4.3, except forming an uniform family, and will be shown to be unstable. Interestingly, the system *is* globally rapid mixing, showing that without the correct structure with respect to system size scaling, rapid mixing alone is not sufficient to imply stability of local observables. This example is the generalization to dissipative systems of the globally gapped but not locally gapped example in [40]. We will show that the characteristics of the dynamics are essentially determined by a classical Markov chain embedded into the Lindbladian. For a general review on convergence of Markov chains, see [33].

Example A.1. Consider a chain of $2N$ classical spins, with values in $\{0,1\}$. Let us define a generator Q^{2N} of a classical Markov chain over the configuration space $\{0,1\}^{2N}$. We will define Q^{2N} in a translational invariant way as follows: call

$$Q_c = \begin{array}{c} \\ |10\rangle \\ |00\rangle \\ |11\rangle \\ |01\rangle \end{array} \begin{array}{c} \begin{array}{cccc} |10\rangle & |00\rangle & |11\rangle & |01\rangle \end{array} \\ \begin{pmatrix} -\frac{2}{3N} & 0 & 0 & \frac{2}{3N} \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}, \qquad Q_r = \begin{array}{c} \\ |10\rangle \\ |00\rangle \\ |11\rangle \\ |01\rangle \end{array} \begin{array}{c} \begin{array}{cccc} |10\rangle & |00\rangle & |11\rangle & |01\rangle \end{array} \\ \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array},$$

$$Q_l = \begin{array}{c} \\ |10\rangle \\ |00\rangle \\ |11\rangle \\ |01\rangle \end{array} \begin{array}{c} \begin{array}{cccc} |10\rangle & |00\rangle & |11\rangle & |01\rangle \end{array} \\ \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}, \qquad \delta_0 = |0\rangle\langle 0|\,, \quad \delta_1 = |1\rangle\langle 1|\,.$$

We then define for each $i = 1 \dots N$, a generator matrix Q_i acting on spins $(2i-2, \dots, 2i+1)$ by

$$Q_i = \mathbb{1} \otimes Q_c \otimes \mathbb{1} + \mathbb{1} \otimes Q_r \otimes \delta_0 + \delta_1 \otimes Q_l \otimes \mathbb{1};$$

and $Q^{2N} = \sum_{i=1}^{N} Q_i$.

The matrix Q_i can only change spins $(2i-1, 2i)$: its transition graph restricted to such spins is presented in figure 2.

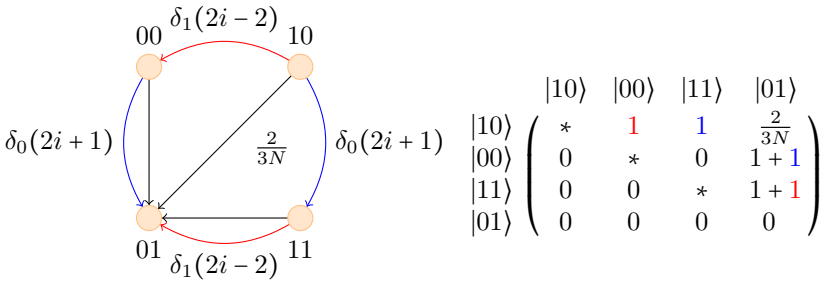

Figure 2: The transition matrix for Q_i on the spins $(2i-1, 2i)$. The blue and the red transitions are present depending on the nearby sites: the blue ones if there is a 0 on the right, the red ones if there is a 1 on the left. Asterisks in the diagonal are such that the sum of each row is zero.

By construction, Q^{2N} is upper triangular. Thus the elements on the diagonal are the eigenvalues. The unique steady state is then $|0101\dots01\rangle$, and the smallest non-zero eigenvalue, corresponding to the state $|1010\dots10\rangle$, is $\frac{2}{3}$. Furthermore, it is easy to see that the diameter of the graph of the transitions of Q^{2N} is N, and in turn this implies that the mixing time for Q^{2N} is of order $O(\log N)$[11].

Let us now embed this classical Markov chain into a Lindbad operator, in a similar fashion as we have done in section 7.1 with Glauber dynamics. We will consider then a chain of $2N$ qubits, and define the following Lindblad operators: if k is odd, then

$$\begin{aligned} L_{k,1} &= \sigma_x^{k+1} |0\rangle\langle 0|_k \otimes |0\rangle\langle 0|_{k+1}\,, \\ L_{k,2} &= \sigma_x^{k} \quad |1\rangle\langle 1|_k \otimes |1\rangle\langle 1|_{k+1}\,, \\ L_{k,3} &= \sqrt{\frac{2}{3N}} \sigma_x^k \otimes \sigma_x^{k+1} |1\rangle\langle 1|_k \otimes |0\rangle\langle 0|_{k+1}\,; \end{aligned}$$

if k is even, then

$$\begin{aligned} L_{k,1} &= \sigma_x^{k} \quad |0\rangle\langle 0|_k \otimes |0\rangle\langle 0|_{k+1}\,, \\ L_{k,2} &= \sigma_x^{k+1} |1\rangle\langle 1|_k \otimes |1\rangle\langle 1|_{k+1}\,, \\ L_{k,3} &= 0. \end{aligned}$$

The Lindbladian is then defined translation-invariantly as

$$\mathcal{L}^{2N} = \sum_{k=1}^{2N} \sum_{i=1}^{3} \mathcal{L}_{k,i} + \mathcal{D}_k;$$

where $\mathcal{D}_k$ is a dephasing channel acting on site k, as in equation (37). Since $L_{k,3}$ depends on N, the family we have defined is not a uniform family.

It is easy to see that the action of $\mathcal{L}^{2N}$ of diagonal states of the form $|\alpha\rangle\langle\alpha|$, with $\alpha \in \{0,1\}^{2N}$, is equal to the one of Q^{2N} acting on α: this is indeed an embedding of Q^{2N}.

[11] This can be seen from the upper triangular form of Q^{2N}, noticing that the polynomials appearing in $e^{tQ^{2N}}$ have degree of at most the diameter of the transition graph.

Then, by a similar argument as in 7.1, we can prove that the fixed points of $\mathcal{L}^{2N}$ are exactly the same of Q^{2N} (namely, the unique state $|0101\ldots01\rangle\langle 0101\ldots01|$), and that the mixing time of $\mathcal{L}^{2N}$ is bounded by the sum of the mixing times of Q^{2N} and of $\mathcal{D}$. Since both of them are mixing in time $O(\log N)$, we see that $\mathcal{L}^{2N}$ verifies global rapid mixing.

But the system is unstable: if we perturb $\mathcal{L}^{2N}$ by removing the terms generated by $L_{k,3}$ (which is a perturbation of order $O(\frac{1}{N})$), the diagonal state $|1010\ldots10\rangle\langle 1010\ldots10|$ becomes a stationary state, and it is clearly locally ortogonal from the original one $|0101\ldots01\rangle\langle 0101\ldots01|$.

References

[1] R. Alicki et al. "On Thermal Stability of Topological Qubit in Kitaev's 4D Model". In: *Open Systems & Information Dynamics* 17.01 (2010), pp. 1–20. arXiv:0811.0033 [quant-ph].

[2] A. Aspuru-Guzik and P. Walther. "Photonic quantum simulators". In: *Nature Physics* 8.4 (2012), pp. 285–291.

[3] J. T. Barreiro et al. "Experimental multiparticle entanglement dynamics induced by decoherence". In: *Nature Physics* 6 (Dec. 2010), pp. 943–946. arXiv:1005.1965 [quant-ph].

[4] R. Blatt and C. Roos. "Quantum simulations with trapped ions". In: *Nature Physics* 8.4 (2012), pp. 277–284.

[5] I. Bloch, J. Dalibard, and S. Nascimbène. "Quantum simulations with ultracold quantum gases". In: *Nature Physics* 8.4 (2012), pp. 267–276.

[6] S. Bravyi, M. B. Hastings, and S. Michalakis. "Topological quantum order: Stability under local perturbations". In: *Journal of Mathematical Physics* 51.9 (Sept. 2010), p. 093512. arXiv:1001.0344 [quant-ph].

[7] H. Briegel et al. "Measurement-based quantum computation". In: *Nature Physics* 5.1 (2009), pp. 19–26.

[8] E. Dennis et al. "Topological quantum memory". In: *Journal of Mathematical Physics* 43 (Sept. 2002), pp. 4452–4505. eprint: arXiv:quant-ph/0110143.

[9] L.-M. Duan and C. Monroe. "*Colloquium*: Quantum networks with trapped ions". In: *Rev. Mod. Phys.* 82 (2 Apr. 2010), pp. 1209–1224.

[10] E. Farhi et al. "A Quantum Adiabatic Evolution Algorithm Applied to Random Instances of an NP-Complete Problem". In: *Science* 292.5516 (2001), pp. 472–475.

[11] V. Gorini, A. Kossakowski, and E. C. G. Sudarshan. "Completely positive dynamical semigroups of N-level systems". In: *J. Mathematical Phys.* 17.5 (1976), pp. 821–825. ISSN: 0022-2488.

[12] L. Gross. "Hypercontractivity, logarithmic Sobolev inequalities, and applications: a survey of surveys". In: *Diffusion, quantum theory, and radically elementary mathematics*. Vol. 47. Math. Notes. Princeton, NJ: Princeton Univ. Press, 2006, pp. 45–73.

[13] L. Gross. "Logarithmic Sobolev inequalities". In: *Amer. J. Math.* 97.4 (1975), pp. 1061–1083. ISSN: 0002-9327.

[14] L. Gross. "Logarithmic Sobolev inequalities and contractivity properties of semigroups". In: *Dirichlet forms (Varenna, 1992)*. Vol. 1563. Lecture Notes in Math. Berlin: Springer, 1993, pp. 54–88.

[15] J. Haah. "Local stabilizer codes in three dimensions without string logical operators". In: *Phys. Rev. A* 83.4, 042330 (Apr. 2011), p. 042330. arXiv:1101.1962 [quant-ph].

[16] K. Hammerer, A. S. Sørensen, and E. S. Polzik. "Quantum interface between light and atomic ensembles". In: *Rev. Mod. Phys.* 82 (2 Apr. 2010), pp. 1041–1093.

[17] M. B. Hastings. "An area law for one-dimensional quantum systems". In: *Journal of Statistical Mechanics: Theory and Experiment* 2007.08 (2007), P08024. arXiv:0705.2024 [quant-ph].

[18] M. B. Hastings. "Lieb-Schultz-Mattis in higher dimensions". In: *Phys. Rev. B* 69 (10 Mar. 2004), p. 104431. arXiv:cond-mat/0305505.

[19] M. B. Hastings. "Locality in Quantum Systems". In: *ArXiv e-prints* (Aug. 2010). arXiv:1008.5137 [math-ph].

[20] M. B. Hastings and X.-G. Wen. "Quasiadiabatic continuation of quantum states: The stability of topological ground-state degeneracy and emergent gauge invariance". In: *Phys. Rev. B* 72 (4 July 2005), p. 045141.

[21] M. B. Hastings and T. Koma. "Spectral Gap and Exponential Decay of Correlations". English. In: *Communications in Mathematical Physics* 265 (3 2006), pp. 781–804. ISSN: 0010-3616. arXiv:math-ph/0507008.

[22] A. A. Houck, H. E. Türeci, and J. Koch. "On-chip quantum simulation with superconducting circuits". In: *Nature Physics* 8.4 (2012), pp. 292–299.

[23] N. Johnston, D. W. Kribs, and V. I. Paulsen. "Computing stabilized norms for quantum operations via the theory of completely bounded maps". In: *Quantum Inf. Comput.* 9.1-2 (2009), pp. 16–35. ISSN: 1533-7146. arXiv:0711.3636 [quant-ph].

[24] S. P. Jordan, K. S. Lee, and J. Preskill. "Quantum algorithms for quantum field theories". In: *Science* 336.6085 (2012), pp. 1130–1133.

[25] M. J. Kastoryano and J. Eisert. *Rapid mixing implies exponential decay of correlations*. 2013. arXiv:1303.6304 [quant-ph].

[26] M. J. Kastoryano et al. *Quantum logorithmic Sobolev inequalities and rapid mixing*. Talk given at QIP 2013, Beijing. URL: http://conference.iiis.tsinghua.edu.cn/QIP2013/wp-content/uploads/2012/01/Michael-Kastoryano.pdf.

[27] M. J. Kastoryano, D. Reeb, and M. M. Wolf. "A cutoff phenomenon for quantum Markov chains". In: *J. Phys. A* 45.7 (2012), pp. 075307, 16. ISSN: 1751-8113. arXiv:1111.2123 [quant-ph].

[28] C. King. *Hypercontractivity for semigroups of unital qubit channels*. Oct. 2012. arXiv:1210.8412 [quant-ph].

[29] A. Kitaev. "Fault-tolerant quantum computation by anyons". In: *Annals of Physics* 303.1 (2003), pp. 2 –30. ISSN: 0003-4916. arXiv:quant-ph/9707021.

[30] I. Klich. *On the stability of topological phases on a lattice*. 2010. URL: http://www.sciencedirect.com/science/article/pii/S000349161000093X.

[31] B. Kraus et al. “Preparation of entangled states by quantum Markov processes”. In: *Phys. Rev. A* 78.4, 042307 (Oct. 2008). arXiv:0803.1463 [quant-ph].

[32] H. Krauter et al. “Entanglement Generated by Dissipation and Steady State Entanglement of Two Macroscopic Objects”. In: *Phys. Rev. Lett.* 107 (8 Aug. 2011), p. 080503.

[33] D. A. Levin, Y. Peres, and E. L. Wilmer. *Markov chains and mixing times*. With a chapter by James G. Propp and David B. Wilson. Providence, RI: American Mathematical Society, 2009, pp. xviii+371. ISBN: 978-0-8218-4739-8.

[34] T. M. Liggett. *Interacting particle systems*. Classics in Mathematics. Reprint of the 1985 original. Berlin: Springer-Verlag, 2005, pp. xvi+496. ISBN: 3-540-22617-6.

[35] G. Lindblad. “On the generators of quantum dynamical semigroups”. In: *Comm. Math. Phys.* 48.2 (1976), pp. 119–130. ISSN: 0010-3616.

[36] E. Lubetzky and A. Sly. “Cutoff for the Ising model on the lattice”. In: *Invent. Math.* 191.3 (2013), pp. 719–755. ISSN: 0020-9910. arXiv:0909.4320 [math.PR].

[37] F. Martinelli, E. Olivieri, and R. H. Schonmann. “For 2-D lattice spin systems weak mixing implies strong mixing”. In: *Comm. Math. Phys.* 165.1 (1994), pp. 33–47. ISSN: 0010-3616.

[38] F. Martinelli. “Lectures on Glauber dynamics for discrete spin models”. In: *Lectures on probability theory and statistics (Saint-Flour, 1997)*. Vol. 1717. Lecture Notes in Math. Berlin: Springer, 1999, pp. 93–191.

[39] P. C. Maurer et al. “Room-Temperature Quantum Bit Memory Exceeding One Second”. In: *Science* 336.6086 (2012), pp. 1283–1286.

[40] S. Michalakis and J. Pytel. *Stability of Frustration-Free Hamiltonians*. Sept. 2011. arXiv:1109.1588 [quant-ph].

[41] B. Nachtergaele, A. Vershynina, and V. A. Zagrebnov. “Lieb-Robinson bounds and existence of the thermodynamic limit for a class of irreversible quantum dynamics”. In: *Entropy and the quantum II*. Vol. 552. Contemp. Math. Providence, RI: Amer. Math. Soc., 2011, pp. 161–175. arXiv:1103.1122 [math-ph].

[42] C. Nayak et al. “Non-Abelian anyons and topological quantum computation”. In: *Rev. Mod. Phys.* 80 (3 Sept. 2008), pp. 1083–1159. arXiv:0707.1889 [cond-mat.str-el].

[43] M. A. Nielsen and I. L. Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010.

[44] D. Poulin. “Lieb-Robinson Bound and Locality for General Markovian Quantum Dynamics”. In: *Phys. Rev. Lett.* 104.19, 190401 (May 2010), p. 190401. arXiv:1003.3675 [quant-ph].

[45] S. Sachdev. *Quantum Phase Transitions*. John Wiley & Sons, Ltd, 2007. ISBN: 9780470022184.

[46] O. Szehr, D. Reeb, and M. M. Wolf. *Spectral convergence bounds for classical and quantum Markov processes*. 2013. arXiv:1301.4827 [quant-ph].

[47] O. Szehr and M. M. Wolf. *Perturbation Bounds for Quantum Markov Processes and their Fixed Points*. 2012. arXiv:1210.1171 [quant-ph].

[48] K. Temme et al. "The χ^2-divergence and mixing times of quantum Markov processes". In: *Journal of Mathematical Physics* 51.12, 122201 (2010), p. 122201. arXiv:1005.2358 [quant-ph].

[49] F. Verstraete, M. M. Wolf, and J. I. Cirac. "Quantum computation and quantum-state engineering driven by dissipation". In: *Nature. Physics* 5.9 (2009), pp. 633–636. arXiv:0803.1447 [quant-ph].

[50] M. M. Wolf and D. Perez-Garcia. "The inverse eigenvalue problem for quantum channels". In: *ArXiv e-prints* (May 2010). arXiv:1005.4545 [quant-ph].

[51] M. M. Wolf. *Quantum Channels & Operations. Guided Tour*. eprint: http://www-m5.ma.tum.de/foswiki/pub/M5/Allgemeines/MichaelWolf/QChannelLecture.pdf.