The structure of degradable quantum channels

Toby S. Cubitt

Department of Mathematics, University of Bristol, University Walk, Bristol BS8 1TW, UK

toby.cubitt@bristol.ac.uk

Mary Beth Ruskai

Department of Mathematics, Tufts University, Medford, MA 02155, USA

marybeth.ruskai@tufts.edu

Graeme Smith

IBM T.J. Watson Research Center, Yorktown Heights, NY 10598, USA

graemesm@us.ibm.com

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Abstract

Degradable quantum channels are among the only channels whose quantum and private classical capacities are known. As such, determining the structure of these channels is a pressing open question in quantum information theory. We give a comprehensive review of what is currently known about the structure of degradable quantum channels, including a number of new results as well as alternate proofs of some known results. In the case of qubits, we provide a complete characterization of all degradable channels with two dimensional output, give a new proof that a qubit channel with two Kraus operators is either degradable or anti-degradable and present a complete description of anti-degradable unital qubit channels with a new proof.

For higher output dimensions we explore the relationship between the output and environment dimensions (d_B and d_E respectively) of degradable channels. For several broad classes of channels we show that they can be modeled with a environment that is "small" in the sense $d_E \leq d_B$. Such channels include all those with qubit or qutrit output, those that map some pure state to an output with full rank, and all those which can be represented using simultaneously diagonal Kraus operators, even in a non-orthogonal basis. Perhaps surprisingly, we also present examples of degradable channels with "large" environments, in the sense that the minimal dimension $d_E > d_B$. Indeed, one

can have $d_E > \frac{1}{4}d_B^2$. These examples can also be used to give a negative answer to the question of whether additivity of the coherent information is helpful for establishing additivity for the Holevo capacity of a pair of channels.

In the case of channels with diagonal Kraus operators, we describe the subclass which are complements of entanglement breaking channels. We also obtain a number of results for channels in the convex hull of conjugations with generalized Pauli matrices. However, a number of open questions remain about these channels and the more general case of random unitary channels.

Contents

1 Introduction

In quantum information theory, a quantum channel is represented by a completely positive, trace-preserving (CPT) map Φ on a suitable algebra of operators. Devetak and Shor [11] introduced the concept of a *degradable* channel by combining the classical notion of a degraded broadcast channel with that of the complement of a channel. A degraded broadcast channel is a single-sender two-reciever broadcast channel in which the one receiver can degrade his/her output to simulate the output of the other. Such channels are among the few classical broadcast channels for which the capacity region is known [7, 8]. Similarly, Devetak and Shor showed that degradable channels have additive coherent information, so that their quantum capacity is given by the coherent information for a single use of the channel. Furthermore, Yard, Devetak, and Hayden have shown [41] that the coherent information of a degradable channel is concave as a function of reference state, so that the required optimization can be performed efficiently and the capacity problem for such channels has been completely resolved.

Before going further, we make these notions explicit. In the finite dimensional case any completely positive trace-preserving (CPT) map, $\Phi : M_{d_A} \mapsto M_{d_B}$, can be represented using an auxiliary space $\mathbf{C}_{d_E}$ in the form

$$\Phi(\rho) = \mathrm{Tr}_E \, U\rho U^\dagger \tag{1}$$

where U is a partial isometry satisfying $U^\dagger U = I_{d_A}$. The complementary channel $\Phi^C : M_{d_A} \mapsto M_{d_E}$ can then be defined [11, 17, 23] by taking the partial trace over the output space d_B so that

$$\Phi^C(\rho) = \mathrm{Tr}_B U\rho U^\dagger. \tag{2}$$

Physically, the complementary channel captures the environment's view of the channel, and as such it is not surprising that its consideration is useful for understanding quantum channel capacities.

Devetak and Shor call a channel *degradable* if there is another CPT map Ψ such that

$$\Psi \circ \Phi = \Phi^C. \tag{3}$$

It is natural to call a channel *anti-degradable* if its complement is degradable, i.e., there is a CPT map Ψ such that $\Psi \circ \Phi^C = \Phi$. Although the complement is only defined up to a partial isometry, this does not affect the concept of degradability because this map can be absorbed into the degrading channel Ψ.

The coherent information of a channel Φ with respect to a reference state ρ was originally defined in terms of a purification. Here, we find it more useful to use an equivalent expression involving the complementary channel,

$$I^{\text{coh}}(\Phi, \rho) = S\left(\Phi(\rho)\right) - S\left(\Phi^C(\rho)\right). \tag{4}$$

The coherent information of Φ is the maximum of (4) over reference states,

$$I^{\text{coh}}(\Phi) = \max_{\rho} I^{\text{coh}}(\Phi, \rho). \tag{5}$$

The quantum capacity of a channel is given by

$$Q_C(\Phi) = \lim_{n\to\infty} \frac{1}{n} I^{\text{coh}}(\Phi^{\otimes n}), \tag{6}$$

as anticipated by Lloyd [26] and others [2]. The proof was completed by Shor [33], Devetak [10] and others [12]. When a channel satisfies the additivity condition,

$$I^{\text{coh}}(\Phi^{\otimes n}) = n I^{\text{coh}}(\Phi), \tag{7}$$

the quantum capacity satisfies the simple "single-letter" formula $Q_C(\Phi) = I^{\text{coh}}(\Phi)$. It was shown in [11] that degradable channels satisfy (7). For completeness, we give a proof of this in Appendix A.2.

Though proving (6) was a significant step towards understanding the quantum channel capacity, it is not known how to cast the quantum capacity of a general channel as a finite optimization problem [13, 35]. As a result, little is known about the quantum capacity of even very basic channels, such as the depolarizing channel. Degradable and anti-degradable channels [16, 39] are among the few for which the quantum capacity is known explicitly. Degradable channels also play a central role in finding bounds on the quantum capacity for more general channels. For example, they were used to find good upper bounds on the capacity of the depolarizing channel [36], especially in the low noise regime. Moreover, it was recently shown [34] that for degradable channels, the coherent information is also equal to the private classical capacity, i.e., the capacity for transmitting classical information protected against an eavesdropper in the sense of [9].

It is well-known that an anti-degradable channel must have zero quantum capacity; as noted in [16], this follows from the no-cloning theorem using an argument that goes back to [3]. A simple analytic argument has also been given by Holevo [19]. Using very different terminology, anti-degradable channels were considered implicitly in several earlier papers [4, 6, 29] in which conditions were given for a Pauli channel to be anti-degradable. We provide an alternate formulation and proof of these results. We also show that every entanglement-breaking channel is anti-degradable. Curiously, although the set of degradable channels is not convex, the set of anti-degradable channels is convex, as shown in Appendix A.3.

Although most channels are neither degradable nor anti-degradable, the implications for quantum capacity have generated some interest in identifying those situations in which the degradability condition (3) holds. Earlier work has shown that any channel with simultaneously diagonalizable Kraus operators is degradable [11], as is the amplitude damping channel [16]. It was shown in [39] that any qubit channel with exactly two Kraus operators is either degradable or antidegradable, with specific condtions under which each (or both) of these hold. Conditions for the degradability of bosonic Gaussian channels were studied in [5, 18, 40], but will not be considered here.

Roughly speaking, degradable channels are those for which the complement is noisier than the original channel, in the sense that the degrading map adds noise to the original channel to generate the complement. Since one would expect noisier channels to be associated with larger environments, it is natural to guess that one must have $d_E \leq d_B$. We show that this holds if any pure input has full rank output, as well as in some specific cases. These include channels with output dimension of 2 or 3, as well as channels whose Kraus operators can be simultaneously diagonalized using a pair of left and right invertible matrices, as discussed in Section 5, following ideas introduced in [39]. Therefore, it may be somewhat surprising that we also find a family of counter-examples which demonstrate that one can have degradable channels with $d_E > d_B$ and that this can happen even when $d_A = d_B$.[1]

The rest of the paper is organized as follows. In Section 2, we study the size of the environment, beginning with some notation and elementary observations in Section 2.1. Then in Section 2.2 we prove that under a condition on output rank any degradable channel must satisfy $d_E \leq d_B$. In Section 2.3 we present examples of degradable channels not satisfying this condition for which $d_E > d_B$. In Section 3, we give a complete classification of degradable channels with qubit outputs. For unital channels mapping qubits to qubits, we give necessary and sufficient conditions for anti-degradability equivalent to earlier work of Niu and Griffiths [29] and Cerf [6]. The details and an alternate proof of the results in [39] for qubit channels with Choi rank 2 are presented in Appendix B. In Section 4 we show that degradable channels with qutrit outputs must have $d_E \leq d_B$, but that other results about qubit maps need not extend to qutrits. In Section 5, we study degradability criteria based

[1]Some results along these lines have recently been established independently by Myhr and Lutkenhaus in their study of symmetric extendable states [27]. Their techniques offer a promising direction for further understanding the structure of degradable channels.

on Kraus diagonal conditions, generalizing the results of [11] and extending some of the ideas in [39]. We pay particular attention to channels whose complement is entanglement breaking, and show any such channel is degradable. We also show that any channel whose Kraus operators can be simultaneously diagonalized, even if different non-orthogonal bases are used for the input and output spaces, has $d_E \leq d_B$ and at least one pure input whose output has full rank. In Section 6, we consider degradability conditions for a special type of random unitary channel in which the unitaries are restricted to generalized Pauli matrices. We show that if such a channel is degradable, then the unitaries commute and $d_E \leq d_B$. In Section 7, we make a few additional observations. One concerns degradability in a neighborhood of the identity. We also observe that the channels introduced in Section 2.3 can be used to show that additivity of coherent information for a pair of channels need not imply additivity of the Holveo capacity for the same pair.

We have also included several appendices. Appendix A.1 describes Arvseon's commutant lifting theorem which can be used to define the complement of a channel in more general and abstract settings. Appendix A.2 contains a proof that degradability implies additivity of coherent information, while Appendix A.3 shows that the set of antidegradable channels is convex. Appendix B contains new proofs of some results about qubit channels. Appendix B.1 introduces some notation and summarizes basic facts about qubit channels. An alternate proof of the results in [39] for qubit channels with Choi rank 2 is given in Appendix B.2. Notation and some basic results needed for our formulation and proof of necessary and conditions for a unital qubit channel to be anti-degradable is given in Appendix B.3. This is followed by analysis of the special cases of 3 Kraus operators and depolarizing channels in Appendices B.4 and B.5 respectively. The latter shows explicitly that when $d_E > d_A$ the degrading map need not be unique. Finally, the general case is considered in Appendix B.6.

2 Size of environment

2.1 Preliminaries

We will use the term Choi rank of a channel to mean the rank of its Choi Jamiolkoswski state representative $(\mathcal{I} \otimes \Phi)(|\beta\rangle\langle\beta|)$, where $|\beta\rangle = \frac{1}{\sqrt{d_A}} \sum_{i=1}^{d_A} |i\rangle|i\rangle$. This is the same as the minimal number of Kraus operators, or the size d_E of the smallest pure environment that can generate that noise. Thus one must have $d_E \leq d_A d_B$. (Note that the Choi rank is *not* the same as the usual rank of Φ considered as linear operator on M_d.)

In principle, deciding whether or not a channel is degradable is straightforward.

A necessary condition for degradability is that

$$\ker \Phi \subseteq \ker \Phi^C. \tag{8}$$

Thus, if there is a matrix $A \in \ker \Phi$ which is not in $\ker \Phi^C$, the channel can not be degradable. Otherwise, when $d_B \leq d_A$, it suffices to compute $\Psi = \Phi^C \circ \Phi^{-1}$ on $[\ker \Phi]^\perp$ and test Ψ for complete positivity. In practice, this may not be so straightforward because composition is the matrix product when Φ and Φ^C are represented in some orthonormal bases for M_d in the standard way (using the Hilbert-Schmidt inner product $\mathrm{Tr}\, A^\dagger B$). However, testing for complete positivity requires reshuffling the result into the form $\sum_{jk} |e_j\rangle\langle e_k| \otimes \Psi(|e_j\rangle\langle e_k|)$. Furthermore, when $d_B > d_A$, Φ does not have a right inverse and the degrading map need not be unique. In Appendix B.5 we show that many unital qubit channels which are anti-degradable have a family of degrading maps rather than a unique degrador.

For $\Phi : M_{d_A} \mapsto M_{d_B}$, the cases in which one of d_A, d_B or the Choi rank d_E equal 1 are all easily treated as follows:

- When $d_A = 1$, both Φ and Φ^C have unique outputs which we denote ρ_B and ρ_E respectively. Moreover, $d_B = d_E$ and Φ is both degradable and anti-degradable with degrading map $\Psi : \gamma \mapsto (\mathrm{Tr}\, \gamma)\rho_B$ (or ρ_E) completely noisy.
- When $d_B = 1$, the only possible CPT map is $\Phi = \mathrm{Tr}$ which must have $d_A = d_E$ and Kraus operators $|\phi\rangle\langle e_k|$. Then $\Phi^C = \mathcal{I}$ and Φ is anti-degradable.
- When $d_E = 1$, any CPT map must have the form $\Phi(\rho) = U\rho U^\dagger$ with $U^\dagger U = I_{d_A}$, which implies that U is a partial isometry and $d_A \leq d_B$. Then $\Phi^C(\rho) = \mathrm{Tr}\, \rho$ and Φ is always degradable with degrading map $\Psi = \mathrm{Tr}$.

Implicit in these examples, is the easily verified fact that $\mathcal{I}^C = \mathrm{Tr}$. We also observe that the situations in $d_E = 1$ and $d_B = 1$ are essentially the only ways in which every pure input has a pure output.

Theorem 1 *If $\Phi : M_{d_A} \mapsto M_{d_B}$ maps every pure state to pure state, then either*

(i) $d_A \leq d_B$ and $\Phi(\rho) = U\rho U^\dagger$ with partial isometry U satifying $U^\dagger U = I_{d_A}$ is always degradable with Choi rank $d_E = 1$, or

(ii) $\Phi(\rho) = (\mathrm{Tr}\, \rho)|\phi\rangle\langle\phi|$ for all ρ is the completely noisy channel which maps a states to a single fixed pure state and is anti-degradable.

Proof: Let $U : \mathbf{C}_{d_A d_E} \mapsto \mathbf{C}_{d_B d_E}$ with $U^\dagger U = I_{d_A d_E}$ be the partial isometry associated with the representation (1). If all outputs are pure, then $U|\alpha_k \otimes e\rangle = |\beta_k \otimes \gamma_k\rangle$ for any orthonormal basis $\{\alpha_k\}$ of M_{d_A}. Since U must map orthogonal vectors to orthogonal vectors, $\langle \beta_j, \beta_k\rangle\langle \gamma_j, \gamma_k\rangle = \delta_{jk}$. Write $j \in J^\perp$ when $\langle \gamma_1, \gamma_j\rangle = 0$. Then

$$j \in J^\perp \;\Rightarrow\; \Phi : |\tfrac{1}{\sqrt{2}}(\alpha_1 + \alpha_j)\rangle\langle \tfrac{1}{\sqrt{2}}(\alpha_1 + \alpha_j)| \mapsto \tfrac{1}{2}|\beta_1\rangle\langle\beta_1| + \tfrac{1}{2}|\beta_j\rangle\langle\beta_j| \tag{9}$$

which is pure if and only if $|\beta_j\rangle = |\beta_1\rangle$. Thus, we have $|\beta_j\rangle = |\beta_1\rangle\ \forall j \in J^\perp$. For $j \notin J^\perp$, we must have $\langle\beta_1, \beta_j\rangle = 0$ and

$$\begin{aligned} j \notin J^\perp \ \Rightarrow\ \Phi : & |\tfrac{1}{\sqrt{2}}(\alpha_1 + \alpha_j)\rangle\langle\tfrac{1}{\sqrt{2}}(\alpha_1 + \alpha_j)| \\ & \mapsto \tfrac{1}{2}\big(|\beta_1\rangle\langle\beta_1| + \langle\gamma_1, \gamma_j\rangle|\beta_1\rangle\langle\beta_j| + \langle\gamma_j, \gamma_1\rangle|\beta_j\rangle\langle\beta_1| + |\beta_j\rangle\langle\beta_j|\big) \end{aligned} \tag{10}$$

which gives a pure output if and only if $|\langle\gamma_1, \gamma_j\rangle| = 1$, or, in other words, $|\gamma_j\rangle = e^{i\theta}|\gamma_1\rangle$.

Now, if $J^\perp$ is empty, then Φ is of the form (i). Otherwise, we can assume that $2 \in J^\perp$ and repeat the argument in (10) to conclude that

$$j \notin J^\perp \rightarrow |\langle\gamma_2, \gamma_j\rangle| = 1$$

which gives a contradiction, since $|\gamma_j\rangle$ can not be proportional to two orthogonal vectors. Hence, $\{j \notin J^\perp\}$ is empty and and Φ has the form (ii). **QED**

2.2 Channels with Small Environment

In this section we show that if a degradable channel maps even one pure state to an output with full rank, then the channel can always be modeled using an environment no larger than the output space. We first prove a more general lemma from which this result follows immediately. Although we restrict attention to finite dimensions we write $\mathcal{H}_A$ for $\mathbf{C}_{d_A}$ and $\mathcal{B}(\mathcal{H}_A)$ for M_{d_A}, etc. to emphasize that we consider mappings involving different spaces, even when they happen to have the same dimension.

Lemma 2 *Let $\Phi : \mathcal{B}(\mathcal{H}_A) \mapsto \mathcal{B}(\mathcal{H}_B)$ be a degradable CPT map, and for a pure state $|\psi_j\rangle$ define $B_j = \text{range } \Phi(|\psi_j\rangle\langle\psi_j|)$ and $E_j = \text{range } \Phi^{\mathrm{C}}(|\psi_j\rangle\langle\psi_j|)$. Then $\dim B_j = \dim E_j$. Moreover, If the vectors $|\psi_1\rangle, |\psi_2\rangle, \ldots |\psi_m\rangle$ have the property $\text{span} \cup_j B_j = \mathcal{H}_B$, then $\text{span} \cup_j E_j = \mathcal{H}_E$.*

Proof: We can write the spectral decomposition of each output as

$$\Phi(|\psi_j\rangle\langle\psi_j|) = \sum_{k=1}^{r_j} \mu_{jk}^2 |\phi_k^j\rangle\langle\phi_k^j| \tag{11}$$

with all $\mu_{jk} > 0$ and $|\phi_k^j\rangle \in \mathcal{H}_B$ orthonormal for each fixed j, i.e., $\langle\phi_k^j, \phi_\ell^j\rangle = \delta_{k\ell}$. By standard purification arguments, it follows that if $U : \mathcal{H}_A \mapsto \mathcal{H}_{BE}$ is the partial isometry in the representation (1) for Φ, one can also find, for fixed j, orthonormal $|\omega_k^j\rangle \in \mathcal{B}(\mathcal{H}_E)$ such that

$$U|\psi_j\rangle = \sum_{k=1}^{r_j} \mu_{jk}\, |\phi_k^j \otimes \omega_k^j\rangle. \tag{12}$$

Note that this implies $r_j = \dim B_j = \dim E_j$. Now let $\Psi : \mathcal{B}(\mathcal{H}_B) \mapsto M_{d_E}$ be the degrading map with environment $\mathcal{H}_G$ whose representation (1) has the operator $V : \mathcal{H}_B \mapsto \mathcal{H}_{EG}$ so that $V : |\phi\rangle \mapsto |\sigma\rangle \in \mathcal{H}_{EG}$. Define $\gamma_k^j \equiv \mathrm{Tr}_G\, V|\phi_k^j\rangle\langle\phi_k^j|V^\dagger$. Then the degradability hypothesis implies that for each j

$$\sum_{k=1}^{r_j} \mu_{jk}^2 |\omega_k^j\rangle\langle\omega_k^j| = \Phi^C(|\psi_j\rangle\langle\psi_j|) = \Psi\circ\Phi(|\psi_j\rangle\langle\psi_j|) = \sum_{k=1}^{r_j} \mu_{jk}^2 \gamma_k^j. \tag{13}$$

Now, If $\mathrm{span} \cup_\mathrm{j} \mathrm{E_j} \neq \mathcal{B}(\mathcal{H}_\mathrm{E})$ then there is a vector $|\omega^\perp\rangle \in \mathcal{B}(\mathcal{H}_E)$ orthogonal to $\mathrm{span}\{|\omega_\mathrm{k}^\mathrm{j}\rangle : \mathrm{j} = 1\ldots\mathrm{m}, \mathrm{k} = 1\ldots\mathrm{r_j}\}$ defined in (12). But then it follows from (13) that

$$\begin{aligned} 0 &= \sum_{j=1}^{m}\sum_{k=1}^{r_j} \mu_{jk}^2 |\langle\omega^\perp, \omega_k^j\rangle|^2 = \sum_{j=1}^{m}\sum_{k=1}^{r_j} \mu_{jk}^2 \mathrm{Tr}\,\gamma_k^j |\omega^\perp\rangle\langle\omega^\perp| \\ &= \sum_{j=1}^{m}\sum_{k=1}^{r_j} \mu_{jk}^2 \langle\omega^\perp \gamma_k^j \omega^\perp\rangle \end{aligned} \tag{14}$$

But since $\mu_{jk}^2 > 0$ for all j,k and each γ_k^j is positive semi-definite, this implies that $\langle\omega^\perp\gamma_k^j\omega^\perp\rangle = 0$ for all j,k. Therefore,

$$\begin{aligned} 0 &= \mathrm{Tr}_E \mathrm{Tr}\, |\omega^\perp\rangle\langle\omega^\perp|\gamma_k^j = \mathrm{Tr}_{EG}(|\omega^\perp\rangle\langle\omega^\perp| \otimes I_G) V|\phi_k^j\rangle\langle\phi_k^j|V^\dagger \\ &= \mathrm{Tr}\,\left[\left(|\omega^\perp\rangle\langle\omega^\perp| \otimes I_G\right) V|\phi_k^j\rangle\langle\phi_k^j|V^\dagger \left(|\omega^\perp\rangle\langle\omega^\perp| \otimes I_G\right)\right] \\ &= \left\|(|\omega^\perp\rangle\langle\omega^\perp| \otimes I_G)V|\phi_k^j\rangle\right\|^2 \end{aligned} \tag{15}$$

so that $(|\omega^\perp\rangle\langle\omega^\perp| \otimes I_G)V|\phi_k^j\rangle = 0$ for all j,k. Since the hypothesis $\mathrm{span}\cup_\mathrm{j}\mathrm{B_j} = \mathcal{B}(\mathcal{H}_\mathrm{B})$ implies that any $|\phi\rangle \in \mathcal{B}(\mathcal{H}_B)$ can be written as a superposition of $|\phi_k^j\rangle$, it follows that $\langle\omega^\perp|\Psi(|\phi\rangle\langle\phi|)|\omega^\perp\rangle = 0$ for any $|\phi\rangle \in \mathcal{B}(\mathcal{H}_B)$. Hence $\mathcal{B}(\mathcal{H}_E) = \mathrm{span}\cup_\mathrm{j}\mathrm{E_j}$. **QED**

Theorem 3 *Let $\Phi : \mathcal{B}(\mathcal{H}_A) \mapsto \mathcal{B}(\mathcal{H}_B)$ be a CPT map with the property that it has at least one pure state whose image $\rho = \Phi(|\psi\rangle\langle\psi|)$ has full rank, i..e, rank $\Phi(\rho) = d_B$. Then if Φ is degradable, $d_E = d_B$.*

Proof: In this case, the hypothesis of Lemma 2 is satisfied with $m = 1$ so that $\mathcal{H}_E = \mathrm{range}\ \Phi^\mathrm{C}(|\psi\rangle\langle\psi|)$. **QED**

For $d_B > 2$, it is not hard to find examples of channels $\Phi : M_{d_A} \mapsto M_{d_B}$ for which no pure input has an output of rank d_B, so that Theorem 3 does not apply. Simply consider a channel which is a convex combination of strictly fewer than d_B unitary conjugations, i.e., $\Phi(\rho) = \sum_{k=1}^{\kappa} U_k\rho U_k^\dagger$ with $\kappa < d_B$. While it was shown by Devetak and Shor that any such channel is degradable when $\kappa = 2$, the question of degradability is unresolved in general when $\kappa > 2$. However, partial results are given in Section 6.

Another example of a channel which has no outputs with full rank is the Werner-Holevo channel $\mathcal{W}(\rho) = \frac{1}{d-1}(I - \rho^T)$, for which every pure input has output of rank exactly $d-1$. For $d = 3$, $\mathcal{W} = \mathcal{W}^C$ so that this channel is both degradable and anti-degradable, as well as an extreme point of the set of CPT maps.

For output dimension $d_B = 2, 3$ one always has $d_E \leq d_B$ as observed in part (i) of Theorem 4 for $d_B = 2$ and proved in Section 4 for $d_B = 3$.

2.3 Degradable channels with large environment

We now give an example which shows that one can have $d_E > d_B$ when $d_B = 2d_A$. Let $\mathcal{N} : M_d \mapsto M_d$ be a CPT map and define $\Phi : M_d \mapsto M_2 \otimes M_d \simeq M_{2d}$ to be the channel

$$\Phi(\rho) = \tfrac{1}{2}|0\rangle\langle 0| \otimes \mathcal{I}(\rho) + \tfrac{1}{2}|1\rangle\langle 1| \otimes \mathcal{N}(\rho) = \rho \oplus \mathcal{N}(\rho) \tag{16}$$

where $\mathcal{I}$ denotes the identity channel . Then

$$\Phi^C(\rho) = \tfrac{1}{2}|0\rangle\langle 0| \otimes \mathrm{Tr}\,\rho + \tfrac{1}{2}|1\rangle\langle 1| \otimes \mathcal{N}^C(\rho) = \tfrac{1}{2}\mathrm{Tr}\,\rho \oplus \tfrac{1}{2}\mathcal{N}^C(\rho). \tag{17}$$

A map $\Psi : M_{2d} \mapsto M_{2d}$ can be defined by its action on product states and extended by linearity. If

$$\Psi(\tau \otimes \gamma) = \langle 0, \tau\, 0\rangle \mathcal{N}^C(\gamma) \oplus \langle 1, \tau\, 1\rangle \mathrm{Tr}\,\gamma, \tag{18}$$

then it is easy to verify that $\Psi \circ \Phi = \Phi^C$ so that Φ is degradable. When $\mathcal{N}$ has d_F Kraus operators A_k so that $\mathcal{N}(\rho) = \sum_k A_k \rho A_k^\dagger$, then the Kraus operators for Φ are $|0\rangle \otimes I$ and $|1\rangle \otimes A_k$ so that it can be represented with a $d_F + 1$ dmiensional environment. In particular, when $\mathcal{N}$ requires the maximum $d_F = d^2$ operators, $d_B = 2d < d^2 + 1 = d_G$; therefore, we have a degradable channel $\Phi : M_{d_A} \mapsto M_{d_B}$ whose environment G has larger dimension than its output space.

One can generalize the channel (16) as follows. For any $x \geq \frac{1}{2}$, and any channel $\mathcal{N} : M_d \to M_d$, we can construct a degradable channel

$$\Phi(\rho) = x|0\rangle\langle 0| \otimes \mathcal{I}(\rho) + (1-x)|1\rangle\langle 1| \otimes \mathcal{N}. \tag{19}$$

The complementary channel is then

$$\Phi^C(\rho) = x|0\rangle\langle 0| \otimes \mathrm{Tr}\,(\rho) + (1-x)|1\rangle\langle 1| \otimes \mathcal{N}^C, \tag{20}$$

to which Φ can be degraded using a channel Ψ whose action on product states is

$$\Psi(\tau \otimes \gamma) = \tfrac{1-x}{x}\langle 0, \tau\, 0\rangle \mathcal{N}^C(\gamma) \oplus \left[\tfrac{2x-1}{x}\langle 0, \tau\, 0\rangle \mathrm{Tr}\,\gamma, +\langle 1, \tau\, 1\rangle \mathrm{Tr}\,\gamma\right]. \tag{21}$$

In this case, it may be clearer to note that this implies

$$\Psi(\gamma_0 \oplus \gamma_1) = \tfrac{1-x}{x} \mathcal{N}^C(\gamma_0) \oplus \left[\tfrac{2x-1}{x} \mathrm{Tr}\, \gamma_0, +\mathrm{Tr}\, \gamma_1\right]. \tag{22}$$

and with a slight abuse of notation corresponds to a channel with Kraus operators

$$|0\rangle\langle 1|\langle j|, \qquad \sqrt{\tfrac{1-x}{x}} |1\rangle\langle 0| \otimes C_i, \qquad \sqrt{\tfrac{2x-1}{x}} |0\rangle\langle 0|\langle j|,$$

where C_i are the Kraus operators of $\mathcal{N}^C$ and $j = 0 \dots d-1$.

It is natural to ask if one must have $d_E \leq d_B$ when $d_A = d_B$? The answer is no, as shown be the following example. Let $d_A = 6$ and $d_B = d_E = 3$. Let $V : \mathbf{C}_6 \mapsto \mathbf{C}_9$ be a partial isometry whose range is the symmetric subspace of $\mathbf{C}_3 \otimes \mathbf{C}_3$ and define a channel $\Phi_2 : M_6 \mapsto M_3$ by

$$\Phi_2(\rho) = \mathrm{Tr}_E V \rho V^\dagger. \tag{23}$$

Since V maps onto the symmetric subspace of $\mathcal{H}_B \otimes \mathcal{H}_E$, $\Phi_2^C(\rho) = \mathrm{Tr}\,_B V \rho V_\dagger = \Phi_2(\rho)$, so that this channel is both degradable and anti-degradable. Now let Φ_1 denote the channel defined in (16) and let $\Phi = \Phi_1 \otimes \Phi_2$. Then Φ is degradable and has $d_A = d_B = 6d$ but $d_E = 3(d^2+1) > 6d = d_B$.

An alternative generalization of (16) is obtained by constructing degradable channels from pairs of channels $\mathcal{M}, \mathcal{N}$ for which there exist channels $\mathcal{X}, \mathcal{Y}$ such that

$$\mathcal{X} \circ \mathcal{N} = \mathcal{M}^C, \qquad \mathcal{Y} \circ \mathcal{M} = \mathcal{N}^C, \tag{24}$$

by letting

$$\Phi(\rho) = \tfrac{1}{2}|0\rangle\langle 0| \otimes \mathcal{M}(\rho) + \tfrac{1}{2}|1\rangle\langle 1| \otimes \mathcal{N}(\rho) = \mathcal{M}(\rho) \oplus \mathcal{N}(\rho). \tag{25}$$

When the environments of $\mathcal{M}$ and $\mathcal{N}$ have dimensions d_E and d_F respectively, the environment of Φ has dimension $d_G = d_E + d_F$. In the example above, $\mathcal{M} = \mathcal{I}$ is universally degradable since one can choose $Y = \mathcal{N}^C$ and its complement $\mathcal{I}^C = \mathrm{Tr}$ is a universal degrador because $\mathrm{Tr}\, \mathcal{N}(\rho) = \mathrm{Tr}\, \rho$. It is an open question whether or not other such pairs, which we call "co-degradable" exist. It is plausible that when one map $\mathcal{M}$ has Choi rank d^2, the other map $\mathcal{N}$ must have Choi rank one. Thus, one might seek additional examples in which both $\mathcal{M}, \mathcal{N}$ have Choi rank $< d^2$. It would be interesting to know the optimal dimensions for pairs of co-degradable channels.

3 Channels with qubit outputs

We now consider channels with qubit outputs. Wolf and Perez-Garcia [39] showed that every CPT map $\Phi : M_2 \mapsto M_2$ with Choi rank ≤ 2 is either degradable or anti-degradable. We present an alternate proof of their result which exploits the representation of qubit channels introduced in [24] and used in [31]. We also show below that no channel with qubit output and Choi rank larger than 2 can be degradable. Therefore, the degradable qubit channels given in [39] in fact exhaust all the possibile degradable qubit channels. The question remains whether there are degradable channels with qubit outputs, but higher dimensional inputs. We show that this can happen only for input dimension 3 and, furthermore, up to unitary conjugations of the input and output, such a channel is unique.

Theorem 4 *Let $\Phi : M_{d_A} \mapsto M_2$ be a CPT map with qubit output. If Φ is degradable,*

(i) its Choi rank d_E is at most two, and

(ii) its input dimension $d_A \leq 3$.

Moreover, when $d_A = 3$, up to unitary conjugations on the input and output,

$$\Phi(\rho) = A_0 \rho A_0^\dagger + A_1 \rho A_1^\dagger \tag{26}$$

with

$$A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^{-1/2} & 0 \end{pmatrix} \qquad A_1 = \begin{pmatrix} 0 & 2^{-1/2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{27}$$

and this channel is both degradable and anti-degradable.

Proof: Part (i) follows from Theorem 3 together with Theorem 1. In particular, by Theorem 3 if we are to have $d_E > 2$, every pure state must be mapped to a rank 1 output. However, in this case the degradability requirement together with Theorem 1 gives $d_E = 1$.

To prove(ii), observe that part (i) implies that we can write

$$\Phi(\rho) = A\rho A^\dagger + B\rho B^\dagger, \tag{28}$$

with $A^\dagger A + B^\dagger B = I_{d_A}$. Without loss of generality, we may choose

$$A = \begin{pmatrix} \sqrt{a_1} & 0 & 0 & \dots & 0 \\ 0 & \sqrt{a_2} & 0 & \dots & 0 \end{pmatrix} \tag{29}$$

so that

$$B^\dagger B = I_{d_A} - A^\dagger A = \begin{pmatrix} 1-a_1 & 0 & 0 & \dots & 0 \\ 0 & 1-a_2 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \tag{30}$$

But since B is a $2 \times d_A$ matrix, $B^\dagger B$ can have rank at most two. Thus we have a contradiction unless $d_A \leq 4$. When $d_A = 4$, we must also have $a_1 = a_2 = 1$. To see that Φ can not be degradable for $d_A = 4$, use the isomorphism $\mathbf{C}_4 \simeq \mathbf{C}_2 \otimes \mathbf{C}_2$ and rewrite all matrices in block form so that $A = \begin{pmatrix} I & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & I \end{pmatrix}$ and ρ has blocks P_{jk}. Then

$$\Phi(\rho) = \begin{pmatrix} P_{11} & 0 \\ 0 & P_{22} \end{pmatrix} \quad \text{but} \quad \Phi^C(\rho) = \begin{pmatrix} \operatorname{Tr} P_{11} & \operatorname{Tr} P_{12} \\ \operatorname{Tr} P_{21} & \operatorname{Tr} P_{22} \end{pmatrix} = \operatorname{Tr}_2 \rho.$$

This will give a contradiction to (8) for a matrix of the form $\begin{pmatrix} 0 & X \\ X^\dagger & 0 \end{pmatrix}$ with $\operatorname{Tr} X \neq 0$. Thus, there are no degradable channels with $d_A = 2$ and $d_B = 4$.

When, $d_A = 3$, either a_1 or a_2 must equal 1, in order to ensure that the rank of $B^\dagger B$ is no greater than 2. Without loss of generality, we can assume that $a_1 = 1$ and denote $a_2 = a$. Then it follows that

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{a} & 0 \end{pmatrix} \qquad \text{and} \qquad B = U \begin{pmatrix} 0 & \sqrt{1-a} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{31}$$

for some unitary U. Now, consider the action of Φ^C on $|0\rangle\langle 0|$ and $|2\rangle\langle 2|$:

$$\begin{aligned} \Phi^C(|0\rangle\langle 0|) &= \begin{pmatrix} \operatorname{Tr} A|0\rangle\langle 0|A^\dagger & \operatorname{Tr} A|0\rangle\langle 0|B^\dagger \\ \operatorname{Tr} B|0\rangle\langle 0|A^\dagger & \operatorname{Tr} B|0\rangle\langle 0|B^\dagger \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = |0\rangle\langle 0|. \end{aligned} \tag{32}$$

$$\begin{aligned} \Phi^C(|2\rangle\langle 2|) &= \begin{pmatrix} \operatorname{Tr} A|2\rangle\langle 2|A^\dagger & \operatorname{Tr} A|2\rangle\langle 2|B^\dagger \\ \operatorname{Tr} B|2\rangle\langle 2|A^\dagger & \operatorname{Tr} B|2\rangle\langle 2|B^\dagger \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = |1\rangle\langle 1|, \end{aligned} \tag{33}$$

and compare it to the action of Φ

$$\Phi(|0\rangle\langle 0|) = |0\rangle\langle 0| \tag{34}$$
$$\Phi(|2\rangle\langle 2|) = U|1\rangle\langle 1|U^\dagger. \tag{35}$$

Since $\Phi^C(|0\rangle\langle 0|)$ and $\Phi^C(|2\rangle\langle 2|)$ are orthogonal, if we hope to degrade Φ to Φ^C, $\Phi(|0\rangle\langle 0|)$ and $\Phi(|2\rangle\langle 2|)$ must also be orthogonal, which is only the case if $U = I$.

To complete the proof we need to show that when Φ is degradable $a = \frac{1}{2}$. Observe that when $U = I$ in (31) Φ satisfies

$$\begin{aligned} &\Phi\left((1-a)|0\rangle\langle 0| - |1\rangle\langle 1| + a|2\rangle\langle 2|\right) \\ &\quad = (1-a)\Phi(|0\rangle\langle 0|) - \Phi(|1\rangle\langle 1|) + a\Phi(|2\rangle\langle 2|) \\ &\quad = (1-a)|0\rangle\langle 0| - a|1\rangle\langle 1| - (1-a)|0\rangle\langle 0| + a|1\rangle\langle 1| = 0, \end{aligned} \tag{36}$$

but

$$\langle 0|\Phi^C\left((1-a)|0\rangle\langle 0| - |1\rangle\langle 1| + a|2\rangle\langle 2|\right)|0\rangle = (1-a) - a = (1-2a). \tag{37}$$

Thus, (8) holds only if $a = \frac{1}{2}$. Finally, observe that when $a = \frac{1}{2}$ it is easy to check that $\Phi = \Phi^C$ so that the channel is both degradable and anti-degradable with degrading map $\Psi = \mathcal{I}$. **QED**

The following theorem is due to Wolf and Perez-Garcia [39]; we present an alternate proof in Appendix B.2. In view of part (i) of Theorem 4, their degradability conditions are necessary as well as sufficient.

Theorem 5 (Wolf and Perez-Garcia) *Up to unitary conjugations on the input and output, the Choi rank two degradable qubit channels are exactly those of the form*

$$\Phi(\rho) = A_+\rho A_+^\dagger + A_-\rho A_-^\dagger, \tag{38}$$

where

$$\begin{aligned} A_+ &= \cos\tfrac{1}{2}v\,\cos\tfrac{1}{2}u\,I + \sin\tfrac{1}{2}v\,\sin\tfrac{1}{2}u\,\sigma_z = \begin{pmatrix} \cos(\frac{1}{2}[v-u]) & 0 \\ 0 & \cos(\frac{1}{2}[u+v]) \end{pmatrix} \\ A_- &= \sin\tfrac{1}{2}v\,\cos\tfrac{1}{2}u\,\sigma_x - i\cos\tfrac{1}{2}v\,\sin\tfrac{1}{2}u\,\sigma_y = \begin{pmatrix} 0 & \sin(\frac{1}{2}[v-u]) \\ \sin(\frac{1}{2}[u+v]) & 0 \end{pmatrix} \end{aligned} \quad , \tag{39}$$

with $|\sin v| \leq |\cos u|$. *Moreover, when* $|\sin v| \geq |\cos u|$, *a channel of the above form is anti-degradable.*

Corollary 6 *The degradable qubit channels* $\Phi : M_2 \to M_2$ *are, up to unitary conjugations on the input and output, exactly those of the form given in Eq. (38) and Eq.(39) with* $|\sin v| \leq |\cos u|$.

Proof: From Theorem 3 we know that any such Φ can have at most two Kraus operators, which with the above theorem implies the result. **QED**

Although degradable qubit maps can not have Choi rank greater than 2, anti-degradable ones can. Moreover, the set of anti-degradable qubit channels is much larger than the expected set of entanglement breaking ones. The set of anti-degradable unital qubit maps was essentially characterized by Cerf [6] and Niu and Griffiths [29] using a rather different language, and without distinguishing the subset of entanglement breaking channels. We give an alternate formulations and proof of their result in Appendix B.6.

Theorem 7 (Cerf, Niu and Griffiths) *A unital qubit channel with Kraus operators $a_k \sigma_k$ with $\sigma_0 = I$, $a_0 \geq a_k \geq 0$ and $\sum_k a_k^2 = 1$ is anti-degradable if and only if*

$$a_i^2 + a_j^2 + a_k^2 + a_i a_j + a_i a_k + a_j a_k \geq \tfrac{1}{2} \tag{40}$$

with i, j, k distinct in $\{1, 2, 3\}$.

It was shown in [24, Appendix A] that the Kraus operators for any unital qubit channel can be chosen to have the form $a_k U \sigma_k V^\dagger$ with U, V unitary and $\sum_k a_k^2 = 1$, Thus, Theorem 7 gives the general result up to unitary conjugations. Although [6] considered only the combination $(i, j, k) = (1, 2, 3)$ with the implicit assumption that a_0^2, the weight given to the identity, was larger than the weight for any other a_k^2, conjugating with some σ_n gives an obvious extension to arbitrary unital qubit channels.

The general result is more easily stated in a representation introduced in [24] in which the action of a unital qubit channel

$$\Phi : \tfrac{1}{2}[I + \sum_j w_k \sigma_k] \longmapsto \tfrac{1}{2}[I + \sum_k \lambda_k w_k \sigma_k)] \tag{41}$$

is given by three multipliers λ_k. (See Appendix B.1). In this framework, Theorem 7 can be restated as follows.

Theorem 8 *A unital qubit channel is anti-degradable if and only if it can be represented using multipliers λ_k satisfying the CP condition $(1 \pm \lambda_k)^2 \geq (\lambda_i \pm \lambda_j)^2$ and the condition*

$$\sum_{k=1}^{3} \Big(1 - |\lambda_k| + \sqrt{(1 - |\lambda_k|)^2 - (|\lambda_i| - |\lambda_j|)^2} \Big) \geq 2 \tag{42}$$

We can summarize the degradability classification of channels with qubit outputs as follows with the understanding that the conditions are given up to unitary transformation on the input and output.

- A channel with $d_B = 2$ is *both* degradable and anti-degradable if the input dimension $d_A = 1$ or $d_A = 3$. When $d_A = 2$, it must also have two Kraus operators and satisfy $\sin u = \cos v$ or, equivalently, $u = v + \frac{\pi}{2}$ in the notation of (85).
- A channel with $d_B = 2$ is degradable (but not anti-degradable) if $d_E = 1$ or if $d_E = 2$ and $\sin u < \cos v$ in the notation of (85).
- A channel with $d_B = 2$ is anti-degradable (but not degradable) if $d_E = 2$ and $\sin u > \cos v$ in the notation of (85).

- A unital channel with $d_A = d_B = 2$ is anti-degradable if it satisfies (40). The subclass which are also EB satisfy $\sum_k |\lambda_k| \leq 1$; however the set of anti-degradable unital qubit channels contains many which are not EB, as described in Appendices B.4 to B.6

In the case of unital qubit channels, these classes also have simple descriptions in the multiplier picture.

4 Channels with output dimension $d_B = 3$

In this section we prove an analogue of part (i) of Theroem 4 for channels with qutrit output. To do this, we will use Lemma 2 to draw conclusions about vectors in the union of the ranges of two pure inputs. We will also need the following complementary lemma to draw conclusion about vectors in the intersection of the ranges of two pure inputs.

Lemma 9 *Let $\Phi : \mathcal{B}(\mathcal{H}_A) \mapsto \mathcal{B}(\mathcal{H}_B)$ be a degradable CPT map, with degrading map $\Psi : \mathcal{B}(\mathcal{H}_B) \mapsto \mathcal{B}(\mathcal{H}_E)$. For a pure state $|\psi\rangle \in \mathcal{H}_A$ define $B_\psi = \text{range } \Phi(|\psi\rangle\langle\psi|)$ and $E_\psi = \text{range } \Phi^{\text{C}}(|\psi\rangle\langle\psi|)$. Then $|\phi\rangle \in B_\psi$ implies $\text{range } \Psi(|\phi\rangle\langle\phi|) \subset \text{E}_\psi$.*

Proof: As in the proof of Lemma 2, (11) and (12) hold (with the subscript j omitted, as it is now redundant). Let $V : \mathcal{H}_B \mapsto \mathcal{H}_{FG}$ be the partial isometry which implements the representation (1) for Ψ so that $\Psi(\rho) = \text{Tr}_G V\rho V^\dagger$. For $|\phi_k\rangle$ the eigenvectors of $\Phi(|\psi\rangle\langle\psi|)$ let $|\sigma_k\rangle = V|\phi_k\rangle$. Then the degradability condition implies

$$\sum_{k=1}^r \mu_k^2 |\omega_k\rangle\langle\omega_k| \quad = \quad \Psi \circ \Phi(|\psi\rangle\langle\psi|) = \sum_{k=1}^r \mu_k^2 \text{Tr}_G\, |\sigma_k\rangle\langle\sigma_k|. \tag{43}$$

Now suppose $|\omega^\perp\rangle$ is orthogonal to E_ψ. Then

$$\begin{aligned} 0 &= \langle\omega^\perp, \sum_{k=1}^r \mu_k^2 |\omega_k\rangle\langle\omega_k|\omega^\perp\rangle = \sum_k \mu_k^2 \langle\omega^\perp, \text{Tr}_G\, (|\sigma_k\rangle\langle\sigma_k|)\omega^\perp\rangle \\ &= \sum_k \mu_k^2\, \text{Tr}_{FG}(|\omega^\perp\rangle\langle\omega^\perp| \otimes I_G)\, |\sigma_k\rangle\langle\sigma_k| \\ &= \sum_k \sum_n \mu_k^2 |\langle\omega^\perp \otimes g_n, \sigma_k\rangle|^2 \end{aligned} \tag{44}$$

where we used $I_G = \sum_n |g_n\rangle\langle g_n|$. Since $\mu_k^2 > 0$, this implies that

$$0 = \langle\omega^\perp \otimes g_n, \sigma_k\rangle = \langle\omega^\perp \otimes g_n, V\phi_k\rangle \tag{45}$$

for all k, n. Now let $|\phi\rangle = \sum_k \alpha_k|\phi_k\rangle$ be an arbitrary vector in B_ψ. Then

$$\langle\omega^\perp \otimes g_n, V|\phi\rangle \;=\; \sum_k \alpha_k\langle\omega^\perp \otimes g_n, V|\phi_k\rangle = 0 \quad \forall\; n \tag{46}$$

so that

$$0 = \mathrm{Tr}\langle\omega^\perp|\mathrm{Tr_G}V|\phi\rangle\langle\phi|V^\dagger|\omega^\perp\rangle = \mathrm{Tr}\,\langle\omega^\perp\Psi(|\phi\rangle\langle\phi|)\omega^\perp\rangle. \tag{47}$$

Since $\omega^\perp$ was an arbitrary vector in $E_\psi^\perp$, this proves that $\Psi(|\phi\rangle\langle\phi|) \subseteq E_\psi$. **QED**

Theorem 10 *Let $\Phi : M_d \mapsto M_3$ be a CPT map with qutrit output. If Φ is degradable, then its Choi rank is at most three.*

Proof: Let $r_{\max} = \max\{\mathrm{rank}\Phi(|\psi\rangle\langle\psi|) : |\psi\rangle \in \mathbf{C}_{d_A}\}$ denote the maximum output rank of the channel over all pure-state inputs in $\mathcal{H}_A$. If $r_{\max} = 3$ the result holds by Theorem 3; and if $r_{\max} = 1$ the result follows from Theorem 1 as for qubits. Thus, we can assume $r_{\max} = 2$. Fix a $|\psi_1\rangle$ such that $r_1 = \mathrm{rank}\;\Phi(|\psi_1\rangle\langle\psi_1|) = 2$. As in Lemma 9, let $B_\psi = \mathrm{range}\;\Phi(|\psi\rangle\langle\psi|)$ and $E_\psi = \mathrm{range}\;\Phi^{\mathrm{C}}(|\psi\rangle\langle\psi|)$. If $B_\psi \subseteq B_1$ for all $|\psi\rangle \in \mathcal{H}_A$, then we have a qubit output embedded in a qutrit space and the result follows from Theorem 4. Otherwise there is a second vector $|\psi_2\rangle$ for which $B_2 \not\subseteq B_1$. If $r_2 = 1$, one can find a superposition $|\psi\rangle = a|\psi_1\rangle + b|\psi_2\rangle$ whose output has rank 2 and for which $B_\psi \not\subseteq B_1$.[2]

Thus we have reduced the problem to the case in which $\dim B_2 = \dim B_1 = 2$, and $B_1 \neq B_2$. The assumption that $d_B = 3$ then implies that $\mathrm{span B_1 \cup B_2} = \mathcal{H}_\mathrm{B}$. Moreover, $\dim \mathcal{H}_B = 3$ implies that $B_1 \cup B_2 \neq \emptyset$. It follows from Lemma 2 that $\dim E_1 = \dim E_2 = 2$ and $\mathrm{span\ E_1 \cup E_2} = \mathcal{H}_\mathrm{E}$. Now let $|\phi\rangle \in B_1 \cap B_2$. By Lemma 9 $\Psi(|\phi\rangle\langle\phi|) \in E_1$ and $\Psi(|\phi\rangle\langle\phi|) \in E_2$. Therefore, $E_1 \cap E_2$ is non-empty, and

$$\dim \mathcal{H}_E = \dim E_1 + \dim E_2 - \dim E_1 \cap E_2 \leq 2 + 2 - 1 = 3. \textbf{ QED} \tag{48}$$

Unlike the case of qubits, not every map $\Phi : M_3 \mapsto M_3$ with Choi rank 3 is either degradable or anti-degradable. A specific class of examples is given in Corollary 16.

[2]To see this write $|\psi_1\rangle = \mu_1|\phi_1 \otimes f_1\rangle + \mu_2|\phi_2 \otimes f_2\rangle$ with ϕ_j and f_j respectively orthogonal for $j = 1, 2$. If $|\psi_2\rangle = |\phi_3 \otimes f_3\rangle$, then $a|\psi_1\rangle + b|\psi_2\rangle$ must have rank ≤ 2, because rank 3 is excluded by assumption. Roughly, the only superposition which could yield a state of rank 1 must have the form $a|\psi_1\rangle - b|\phi_j \otimes f_j\rangle$; however, the assumption $B_2 \not\subseteq B_1$ precludes $|\phi_3\rangle = |\phi_j\rangle$ for $j = 1, 2$. For a precise argument, write $|\psi_2\rangle = t_1|\phi_1 \otimes f_3\rangle + t_2|\phi_2 \otimes f_3\rangle + t_3|\widehat{\phi}_3 \otimes f_3\rangle$ with $\langle\widehat{\phi}_3, \phi_j\rangle = 0\; j = 1, 2$. Let

$$|\psi\rangle \equiv a|\psi_1\rangle + b|\psi_2\rangle = |\phi_1 \otimes g_1\rangle + |\phi_2 \otimes g_2\rangle + |\widehat{\phi}_3 \otimes g_3\rangle$$

with unnormalized vectors $g_j = a\mu_j f_j + bt_j f_3$ for $j = 1, 2$ and $g_3 = bt_3 f_3$. Then the density matrix $\mathrm{Tr}_E|\psi\rangle\langle\psi|$ can be represented by the 3×3 matrix with elements $\langle g_j, g_k\rangle$. If this has rank 1, then the subdeterminants $\langle g_j, g_j\rangle\langle g_k, g_k\rangle - |\langle g_j, g_k\rangle|^2 = 0$. But this implies $g_3 = cg_j$ for $j = 1, 2$ which implies $f_3 = c' f_1$ and $f_3 = c'' f_2$ which is impossible since $\langle f_1, f_2\rangle = 0$.

For $\Phi : M_4 \mapsto M_4$ one can obtain a simpler example. Let Φ_1 be a qubit channel which is degradable (but not anti-degradable) and Φ_2 be a qubit channel which is anti-degradable (but not degradable). Then $\Phi = \Phi_1 \oplus \Phi_2$ has 4 Kraus operators, but is neither degradable nor anti-degradable.

5 Kraus diagonal conditions

Devetak and Shor [11], showed that any channel with simultaneously diagonalizable Kraus operators is degradable. These are often called "diagonal channels" following terminology introduced in [25] and followed in [21]. However, we prefer the term "Hadamard" used in [23] or "Kraus diagonal" to avoid confusion with channels represented by a diagonal matrix when thought of as a linear operator on the vector space of density operators. King [21] showed that a CP map has diagonal Kraus operators if and only if it can be represented in the form $\rho \mapsto H * \rho$ with H positive semi-definite where $*$ denotes Hadamard (or pointwise) multiplication. It is easy to invert $H * \rho$ since $J * H * \rho = \rho$ when J has elements $1/h_{jk}$.

A channel is equivalent to one with diagonal Kraus operators if there are unitary U, V such that $A_m = U^\dagger D_m V$ where D_k is diagonal with elements a_{jm} on the diagonal. Thus, in essence, the operators A_m have a simultaneous SVD in which one has dropped the usual requirement of positive elements on the diagonal. The matrix H then has elements $h_{jk} = \sum_m a_{jm} \overline{a}_{km}$. Thus

$$\Phi(\rho) = \Gamma_{U^\dagger}\big(H * \Gamma_V(\rho)\big) \tag{49}$$

where $\Gamma_V(\rho) = V \rho V^\dagger$.

In [39], Wolf and Perez-Garcia introduced the notion of "twisted diagonal" for $\Phi : M_{d_A} \mapsto M_{d_B}$ with Kraus operators A_m. They considered only $d_A = d_B$ and required that there exist invertible Y, X such that $Y A_m X$ is diagonal. It is not hard to see that this can be extended to channels with $d_A \leq d_B$ for which Y and X have left and right inverses satisfying $Y_L^{-1} Y = I_A$ and $X X_R^{-1} = I_A$ respectively. The main idea is that Φ can then be written as a composition using single conjugations and Hadamard multiplication, i.e, $\Phi(\rho) = \Gamma_Y\big(H * \Gamma_X(\rho)\big)$ where $\Gamma_Y(A) = Y A Y^\dagger$. Since these maps are easy to invert, Wolf and Perez-Garcia could then give a simple test for degradability of twisted diagonal channels. They also showed that a channel $\Phi : M_d \mapsto M_d$ with Choi rank two is twisted diagonal if one of the Kraus operators has rank d_A. The extreme amplitude-damping channel with Kraus operators $|0\rangle\langle 1|$ and $|0\rangle\langle 0|$ is not twisted diagonal because a matrix of the form $A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ can not be further reduced.

A large class of degradable channels that are twisted diagonal can be constructed by considering the complements of entanglement breaking (EB) maps. It is convenient to begin with the map $\Phi^C : M_{d_A} \mapsto M_{d_E}$ and recall that [20] a CP map Φ^C is EB if and only if its Kraus operators can be chosen to have rank one, so that

$$\Phi^C(\rho) = \sum_k A_k \rho A_k^\dagger = \sum_k |x_k\rangle\langle x_k| \langle w_k, \rho\, w_k\rangle \tag{50}$$

with $A_k = |x_k\rangle\langle w_k|$. It was shown in [17, 23] that the complement $\Phi : M_{d_A} \mapsto M_{d_B}$ has the form

$$\Phi(\rho) = \sum_m F_m \rho F_m^\dagger = \sum_{jk} |e_j\rangle\langle e_k| x_{jk} \langle w_j, \rho\, w_k\rangle \tag{51}$$

with $x_{jk} = \langle x_j, x_k\rangle$. Moreover, the Kraus operators of Φ have the pseudo-diagonal form $F_m = \sum_k c_{km} |e_k\rangle\langle w_k|$, where $C = (c_{km})$ satisfies $(CC^\dagger)_{jk} = \langle x_j, x_k\rangle$. We call this pseudo-diagonal because the vectors $|w_k\rangle$ need not be orthonormal, although the $|e_k\rangle$ are orthogonal. Note that if W is the matrix with elements $\langle w_j, \rho\, w_k\rangle$, then $\Phi(\rho)$ is represented by the Hadamard product $X * W$. It was also shown in [17, 23] that a channel has the form (51) if and only if it is the complement of an EB map. A pseudo-diagonal channel is a special case of a twisted diagonal channel with Y unitary. It follows from Theorem 6 in [20] that $d_E \le d_B$. (In our notation d_B is the dimension of the environment of the EB channel Φ^C. Actually, this result is stated only for $d_A = d_E$ but easily generalizes to $d_B \ge \max\{d_A, d_E\}$.)

Theorem 11 *Every pseudo-diagonal channel is degradable. Equivalently, every entanglement breaking channel is anti-degradable.*

Proof: Let Ψ be the CP map with Kraus operators $G_k = \frac{1}{\|x_k\|} |x_k\rangle\langle e_k|$. Then it follows immediately from (51) that

$$\begin{aligned} \Psi \circ \Phi(\rho) &= \sum_\ell \sum_{jk} \delta_{j\ell}\delta_{k\ell}\, |x_\ell\rangle\langle x_\ell|\, \frac{\langle x_j, x_k\rangle}{\langle x_k, x_k\rangle}\, \langle w_j, \rho\, w_k\rangle & (52)\\ &= \sum_\ell |x_\ell\rangle\langle x_\ell|\, \langle w_\ell, \rho\, w_v\rangle = \Phi^C(\rho).\ \textbf{QED} & (53) \end{aligned}$$

Theorem 12 *If $\Phi : M_{d_A} \mapsto M_{d_B}$ is twisted diagonal with $d_A = d_B$, then $d_B \ge d_E$ and there is a pure state such that the rank of the output $\Phi(|\psi\rangle\langle\psi|)$ is d_B.*

Proof: The d_E Kraus operators A_m in a minimal set are linearly independent because they are eigenvectors of the CJ matrix. For $d_A = d_B$ left and right inverses exist if and only if X, Y are invertible. Thus $A_m = Y D_m X$ with X, Y invertible and D_k diagonal with a_{jm} on the diagonal. The vectors $\mathbf{a}_m$ are also linearly independent, which implies that $d_E \leq d_B$. Let $\mathbf{a}_m$ denote the vectors with elements a_{jm} and $H = \sum_m \mathbf{a}_m \mathbf{a}_m^\dagger$, and note that it has rank d_E. Then

$$\Phi(|\psi\rangle\langle\psi|) = Y\big[H * (X|\psi\rangle\langle\psi|X^\dagger)\big]Y^\dagger. \tag{54}$$

Since X is invertible, one can find $|\psi_1\rangle$ such that $X|\psi_1\rangle$ is proportional to $(1, \ldots, 1)^T$. Then $H * (X|\psi\rangle\langle\psi|X^\dagger) = cH$ for some constant c. Since Y is invertible, it does not affect the rank, so $\Phi(|\psi\rangle\langle\psi|)$ has rank d_E. **QED**

It is curious that we could not show directly that there is an input whose output has full rank, and apply Theorem 3. Instead, we first showed that $d_E \leq d_B$ and used this to conclude that a pure state with full rank output exists. In the case of pseudo-diagonal channels, we have also been unable to show that there is a pure input whose output has full rank. It would be enough to show that one can find a ψ such that $\langle w_k, \psi\rangle \neq 0$ for all k.

6 Random unitary and Pauli diagonal channels

We now explore the conditions for the degradability of random unitary channels. A random unitary channel $\Phi : M_d \mapsto M_d$ is a convex combination of unitary conjugations, i.e.,

$$\Phi(\rho) = \sum_{k=1}^{\kappa} a_k U_k \rho U_k^\dagger \tag{55}$$

with each $a_k \geq 0$ and $\sum_k a_k = 1$. When there are precisely κ distinct unitaries, a pure input can have output of rank at most κ. If there are d or more unitaries, one would expect that one can always find at least one pure input whose output has rank d. If so, one can apply Theorem 3. However, we have not found a proof of this, and one can easily construct examples for which some inputs have lower rank. Nevertheless, one can show directly that for an important subclass of random unitary channels, degradability implies $d_E \leq d_B = d_A$.

Let X and Z denote the matrices whose action on the standard basis is $X|e_k\rangle = |e_{k+1}\rangle$ and $Z|e_k\rangle = e^{i2\pi k/d}|e_k\rangle$. The unitary matrices $X^j Z^k$ are called generalized Pauli matrices and give a projective representation of the Weyl-Heisenberg group. Let V_m denote some ordering of $X^j Z^k$ with $V_0 = I$. Then $\operatorname{Tr} V_m^\dagger V_m = d\delta_{mn}$ and one can write any density matrix in M_d as

$$\rho = \tfrac{1}{d}\big[I + \sum_{k=1}^{d^2-1} v_k V_k\big] \tag{56}$$

with $v_m = \operatorname{Tr} V_m^\dagger \rho$. One can show that that $|v_m| \leq 1$ and $\sum_m |v_m|^2 = d-1$. Moreover, when ρ is pure $|v_m| = 1$ for exactly $d-1$ of the v_m and the rest are zero. For details see [14, 23, 28].

We now restrict attention to channels Φ of the form (55) where each unitary is one of the generalized Pauli matrices. Any such channel is equivalent via unitary conjugation to a channel with $a_0 \neq 0$, and we will assume that this holds. In general, if the V_m corresponding to the remaining non-zero a_m do not commute, we do not expect the channel to be degradable. Theorem 13, together with Corollary 14 makes this intuition precise. The channel Φ is represented by the matrix with elements $\operatorname{Tr} V_n^\dagger \Phi(V_m) = \phi_m \delta_{mn}$ where $\phi_m = \sum_n \xi_{mn} a_n$ and $\xi_{mn} = \frac{1}{d} \operatorname{Tr} V_m V_n V_m^\dagger V_n^\dagger$ is a d^{th} root of unity arising from the Weyl-Heisenberg commutation relations. Therefore, $|\phi_m| \leq 1$ with equality if and only if $\xi_{mn} = +1$ whenever $a_n \neq 0$. Since Φ is represented by a diagonal matrix, we call such channels *Pauli diagonal*. The effect of Φ on a density matrix represented in the form (56) is simply to map $v_m \mapsto \phi_m v_m$.

Theorem 13 *Let Φ be a channel of the form* (55) *with each U_k one of the generalized Pauli matrices. If $a_0 \neq 0$ and for some $m > 0$, $a_m \neq 0$ and $0 < |\phi_m| < 1$, then Φ is not degradable.*

Proof: For simplicity, we first consider the case when V_m has order d, i.e., $V_m^d = I$ but $V_m^\kappa \neq I$ for any positive integer $\kappa < d$. Then $\rho = \frac{1}{d} \sum_{k=0}^{d-1} V_m^k$ projects onto an eigenstate of V_m and is, hence, positive semi-definite. We will show that Φ is not degradable by showing that $\Phi^C \circ \Phi^{-1}(\rho)$ is not positive semi-definite. (If some $\phi_n = 0$, then Φ is not degradable unless $\Phi^C(V_n) = 0$ also. When this happens, it suffices to invert Φ on $\ker(\Phi)^\perp$.)

Using an obvious abuse of notation, we find

$$\Phi^{-1}(\rho) = \frac{1}{d} \sum_{k=0}^{d-1} \phi_{V_m^k}^{-1} V_m^k.$$

Now, it suffices to consider the following 2×2 submatrix of $(\Phi^C \circ \Phi^{-1})(\rho)$,

$$\begin{pmatrix} a_0 & \sqrt{a_0 a_m} \operatorname{Tr} \Phi^{-1}(\rho) V_m^\dagger \\ \sqrt{a_0 a_m} \operatorname{Tr} V_m \Phi^{-1}(\rho) & a_m \operatorname{Tr} V_m \Phi^{-1}(\rho) V_m^\dagger \end{pmatrix} = \begin{pmatrix} a_0 & \sqrt{a_0 a_m} \phi_m^{-1} \\ \sqrt{a_0 a_m} \overline{\phi}_m^{-1} & a_m \end{pmatrix}, \quad (57)$$

the determinant of which is $a_0 a_m (1 - |\phi_m|^{-2}) < 0$.

If d is not prime, e.g., $d = d_1 d_2$ and $V_m^{d_2} = I$, then $V_m = V_{m^*}^{d_1}$ for some m^*. In that case, we can apply the same argument to $\rho = \frac{1}{d} \sum_{k=0}^{d} V_{m*}^k$. **QED**

Corollary 14 *Let Φ be a Pauli-diagonal channel with $a_l \neq 0$ and $a_k \neq 0$ and $V_l V_k \neq V_k V_l$. Then Φ is not degradable.*

Proof: Assume, without loss of generality, that $V_0 = I$ and $a_0 \neq 0$. First note that if there is some $a_m \neq 0$ such that $\phi_m = 0$, the channel cannot be degradable, since

$$\Phi(V_m) = 0, \tag{58}$$

but

$$\langle 0|\Phi^C(V_m)|m\rangle = \sqrt{a_m}\,\mathrm{Tr}\, V_m V_m^\dagger = d\sqrt{a_m} \neq 0. \tag{59}$$

But the usual observation that $\mathrm{Ker}\Phi^C \subset \mathrm{Ker}\Phi$ is required for degradability shows that the channel could not be degradable.

If we can also rule out the possibility that $|\phi_m| = 1$ for all m with $a_m \neq 0$, we will be able to use Theorem 13 to establish the result. But, recall that $|\phi_m| = 1$ only if $\xi_{mn} = 1$ for all $a_n \neq 0$, so that in this case all the V_n with nonzero a_n must commute. **QED**

Corollary 15 *Let Φ be a channel of the form* (55) *with each U_k one of the generalized Pauli matrices and Choi rank $> d$, i.e., $|\{a_m \neq 0\}| > d$. Then Φ is not degradable.*

Proof: Since the generalized Paulis are linearly independent, any mutually commuting subset can contain at most d elements, so that there must be at least two V_n with nonzero a_n that don't commute, which by the previous corollary establishes the result. **QED**

The following corollary is of interest because is shows that for $d > 2$, there are channels with exactly d Kraus operators which are neither degradable nor anti-degradable.

Corollary 16 *Let $\Phi : M_3 \mapsto M_3$ be the channel $\Phi(\rho) = a_0\rho + a_1 X\rho X^\dagger + a_2 Z\rho Z^\dagger$ with a_0, a_1, a_2 strictly positive and at least two unequal. Then Φ is neither degradable nor anti-degradable.*

Proof: Since X and Z do not commute, it follows from Corollary 14 that Φ is not degradable. Indeed, this holds even when $a_0 = a_1 = a_2 = \frac{1}{3}$. To show that Φ is not anti-degradable, we show that Φ has strictly positive coherent information $I_Q(\Phi)$ by considering its action on a maximally entangled state $|\beta\rangle$. One finds that $\Phi(|\beta\rangle\langle\beta|) = \Phi(\frac{1}{3}I) = \frac{1}{3}I$ and that $(\Phi \otimes \mathcal{I})(|\beta\rangle\langle\beta|)$ has eigenvalues a_0, a_1, a_2. To see the latter it suffices to observe that the states $\{|\beta\rangle,\ X|\beta\rangle,\ Z|\beta\rangle\}$ are mutually orthogonal. But this holds since, e.g.,

$$\langle\beta, (X \otimes I)\beta\rangle = \mathrm{Tr}_{12}(X \otimes I)|\beta\rangle\langle\beta| = \mathrm{Tr}\, X(\tfrac{1}{3}I) = 0$$

Thus, we find $I_Q(\Phi) \geq \log 3 + \sum_k a_k \log a_k > 0$ unless $a_0 = a_1 = a_2 = \frac{1}{3}$. **QED**

We have not resolved the question of whether or not the channel with all $a_k = \frac{1}{3}$ is anti-degradable. A more interesting question is whether or not the degradability result (which holds even when all $a_k = \frac{1}{3}$) remains true when X is replaced by an arbitrary unitary operator which does not commute with Z.

7 Additional remarks

It was also shown in [39] that *any* channel Φ with Choi rank two that is sufficiently close to the identity map $\mathcal{I}$ is degradable. It is worth remarking this is not the same as $\Phi = (1-\epsilon)\mathcal{I} + \epsilon\Gamma$ with Γ a channel with Choi rank two unless Γ is itself a convex combination of $\mathcal{I}$ and a unitary conjugation. As remarked in [39], their results do not apply to maps with Choi rank > 2 [35], even for qubits. Corollary 16 implies that a channel

$$\Phi(\rho) = (1 - \epsilon_1 - \epsilon_2)\rho + \epsilon_1 X\rho X^\dagger + \epsilon_2 Z\rho Z^\dagger$$

is neither degradable nor anti-degradable no matter how small $\epsilon_1 + \epsilon_2$ is. Thus, there are rank 3 channels with a trit output that are nondegradable, even arbitrarily close to the identity channel.

The channel (16) can be used to make a small observation on one of the major open questions in quantum information theory, namely, whether or not the Holevo capacity,

$$C_{\rm Hv}(\Phi) = \sup_{\pi_k \rho_k} \Big(S\big(\sum_k \pi_k \rho_k\big) - \sum_k \pi_k S(\rho_k) \Big) = S(\rho_{\rm av}) - Av[S(\rho)] \tag{60}$$

is additive under tensor products. There has been some speculation that degradabillity of Φ or, more generally, additivity of the coherent information would imply additivity for (60). That this implication need not hold can be demonstrated using the channel (16). First, note [15, 38] that if $\Phi = \Phi_1 \oplus \Phi_2$, then $C_{\rm Hv}(\Phi) = C_{\rm Hv}(\Phi_1) + C_{\rm Hv}(\Phi_2)$. For the degradable channel (16) this becomes $C_{\rm Hv}(\Phi) = C_{\rm Hv}(\mathcal{N}) + \log d$, and it follows that $C_{\rm Hv}(\Phi)$ is additive if and only if $C_{\rm Hv}(\mathcal{N})$ is additive. Thus, if a counter-example to additivity for $C_{\rm Hv}(\mathcal{N})$ can be found, then $C_{\rm Hv}(\Phi)$ would be superadditive despite the fact that it is degradable.

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A Background

A.1 Arveson commutant lifting theorem

The complement of a channel $\Phi : M_{d_A} \mapsto M_{d_B}$ is closely related to a map from $\Upsilon : M_{d_A}$ to M_{d_E} defined earlier in greater generality by Arveson[1]. We explain this following the notation in Appendix A of [23], where it was observed that the ancilla representation (1) is a special case of Stinespring's fundamental representation theorem [1, 30, 37]. For CPT maps, it is more convenient to write this for the dual $\widehat{\Phi} : M_{d_B} \mapsto M_{d_A}$ which is unital and defined by the relation $\mathrm{Tr}\, [\widehat{\Phi}(X)]^\dagger \gamma = \mathrm{Tr}\, X^\dagger \Phi(\gamma)$. The Stinespring representation then has the form

$$\widehat{\Phi}(Q) = V^\dagger \pi(Q) V \tag{61}$$

where π is a representation of the algebra, and $V^\dagger V = I_A$ so that V is a partial isometry. Arveson's commutant lifting theorem [1] defines a map $\rho \mapsto X$ by the relation

$$XV = V\rho \tag{62}$$

with X in the commutant of $\pi(M_{d_B})$ (or, X in the commutant of $\pi(\mathcal{B})$ in the general case $\Phi : \mathcal{A} \mapsto \mathcal{B}$ of maps on operator algebras.) Then formally, $\Upsilon_\Phi(\rho) = (V\rho V^\dagger)(VV^\dagger)^{-1}$. For matrix algebras, the inverse above is well-defined on $(\ker V^\dagger)^\perp$; however, in the general setting it may require an unbounded operator affiliated with the algebra $\mathcal{B}$.

As explained in [30, Chapter 2], for maps on matrix algebra one can choose the representation as $\pi(Q) = Q \otimes I_E$. Then one can write $V = \sum_j F_j \otimes |j\rangle$ as a vector of block matrices with the blocks F_k the Kraus operators of Φ, and (61) reduces to (1). In the finite dimensional case with the representation chosen to have the simple form above, the matrix X must then have the form $X = I_B \otimes X_E$ and $X_E \Phi^C(I) = \Phi^C(\rho)$.

Since (62) implies $VQV^\dagger = (I_B \otimes X_E)VV^\dagger$, using the block vector expression for V above gives

$$\sum_{jk} F_j Q F_k^\dagger \otimes |j\rangle\langle k| = \sum_{jk} F_j F_k^\dagger \otimes X_E |j\rangle\langle k| \tag{63}$$

Then taking the partial trace over B and using $\mathrm{Tr}\, F_j F_k^\dagger = \delta_{jk}\tau_k$ yields

$$\begin{aligned} \sum_{jk} (\mathrm{Tr}\, F_j Q F_k^\dagger)\, |j\rangle\langle k| &= \sum_{jk} \mathrm{Tr}\,(F_j F_k^\dagger) \otimes X_E |j\rangle\langle k| && (64) \\ &= X_E \Big(\sum_{jk} \mathrm{Tr}\,(F_j F_k^\dagger) |j\rangle\langle k| \Big). && (65) \end{aligned}$$

Since the left side of (64) is exactly the form of $\Phi^C(\rho)$ given by Eq. (6) in [23], we can conclude that

$$\Phi^C(\rho) = X_E \Phi^C(I_A) = \widetilde{\Upsilon}_\Phi(\rho)\Phi^C(I_A) \tag{66}$$

with $\widetilde{\Upsilon}_\Phi(\rho) \equiv X_E = \frac{1}{d_B}\mathrm{Tr}_B\, \Upsilon_\Phi(\rho)$ obtained from Arveson's Theorem.

Although this establishes a relation between the complement of a channel and Arveson's lifting, it might appear that one can only use (66) to obtain Arveson's channel from the complement, but not the reverse. However, one can also do the latter by choosing the F_k to be the eigenvectors of the Choi matrix of Φ after unstacking and renormalized so that $\mathrm{Tr}\, F_k F_k^\dagger = \tau_k$ are the non-zero eigenvalues of the Choi matrix. Then

$$\Phi^C(I_A) = \sum_{k=0}^{d_E} \tau_k |k\rangle\langle k| \equiv D_\Phi \tag{67}$$

is unitarily equivalent to the projection of the Choi matrix of Φ onto the orthogonal complement of its kernel. To see this write the spectral representation of the Choi matrix as $\displaystyle\sum_{k=0}^{d_A d_B} \tau_k |f_k\rangle\langle f_k|$ with $|f_k\rangle$ the normalized eigenvectors corresponding to F_k. Omitting the eigenvectors with $\tau_k = 0$ gives D_Φ. Thus $D_\Phi^{-1} = \sum_k \tau_k^{-1} |k\rangle\langle k|$ is well defined and

$$\Phi^C(\rho) = \widetilde{\Upsilon}_\Phi(\rho) D_\Phi \qquad \text{or} \qquad \widetilde{\Upsilon}_\Phi(\rho) = \Phi^C(\rho) D_\Phi^{-1}. \tag{68}$$

This allows one to obtain either the complement from Arveson's channel or Arveson's channel from the complement.

A.2 Degradability implies additivity

The standard definition of the coherent information of a channel $\Phi : \mathcal{B}(\mathcal{H}_{A_2}) \mapsto \mathcal{B}(\mathcal{H}_B)$ with respect to a reference state ρ is

$$I^{\mathrm{coh}}(\Phi, \rho) = S[\Phi(\rho)] - S[(I \otimes \Phi)(|\chi\rangle\langle\chi|)] \tag{69}$$

with $|\chi\rangle$ in $\mathcal{H}_{A_1A_2} \equiv \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$ satisfying the purification condition $\mathrm{Tr}_{A_1}|\chi\rangle\langle\chi| = \rho$. But by the Stinespring representation

$$(I \otimes \Phi)(|\chi\rangle\langle\chi|) = \mathrm{Tr}_E\,(I \otimes V)|\chi\rangle\langle\chi|(I \otimes V)^\dagger \tag{70}$$

with $V : \mathcal{H}_{A_2E} \mapsto \mathcal{H}_{BE}$ a partial isometry. Now, since $(I \otimes V)|\chi\rangle\langle\chi|(I \otimes V)^\dagger$ is a pure state,

$$\begin{aligned} S[(I \otimes \Phi)(|\chi\rangle\langle\chi|)] &= S\big[\mathrm{Tr}_E\,(I \otimes V)|\chi\rangle\langle\chi|(I \otimes V)^\dagger\big] \\ &= S\big(\mathrm{Tr}_{A_1B}\,(I \otimes V)|\chi\rangle\langle\chi|(I \otimes V)^\dagger\big) \\ &= S(\mathrm{Tr}_B V\rho V^\dagger) = S\big[\Phi^C(\rho)\big] \end{aligned} \tag{71}$$

Inserting this in (69) yields (4).

To show that degradability implies additivity, observe that the monotonicity of relative entropy $H(\rho, \gamma) \equiv \mathrm{Tr}\,\rho(\log\rho - \log\gamma)$ under CPT maps implies

$$\begin{aligned} &-S[\Phi^C(\rho_{AB})] + S[\Phi^C(\rho_A)] + S[\Phi^C(\rho_B)] \\ &\qquad = H\big[(\Phi^C \otimes \Phi^C)(\rho_{AB}), (\Phi^C \otimes \Phi^C)(\rho_A \otimes \rho_B)\big] \\ &\qquad = H\big[(\Psi \otimes \Psi) \circ (\Phi \otimes \Phi)(\rho_{AB}), (\Psi \otimes \Psi) \circ (\Phi \otimes \Phi)(\rho_A \otimes \rho_B)\big] \\ &\qquad \le H\big[(\Phi \otimes \Phi)(\rho_{AB}), (\Phi \otimes \Phi)(\rho_A \otimes \rho_B)\big] \\ &\qquad = -S[\Phi(\rho_{AB})] + S[\Phi(\rho_A)] + S[\Phi(\rho_B)]. \end{aligned}$$

Rearranging gives

$$S[\Phi(\rho_{AB})] - S[\Phi^C(\rho_{AB})] \;\le\; S[\Phi(\rho_A)] - S[\Phi^C(\rho_A)] \;+\; S[\Phi(\rho_B)] - S[\Phi^C(\rho_B)]$$

which by (4) is equivalent to

$$I^{\mathrm{coh}}(\Phi \otimes \Phi, \rho_{AB}) \le I^{\mathrm{coh}}(\Phi, \rho_A) + I^{\mathrm{coh}}(\Phi, \rho_B). \tag{72}$$

This implies $I^{\mathrm{coh}}(\Phi \otimes \Phi) \le 2I^{\mathrm{coh}}(\Phi)$ and the reverse inequality is trivial. This argument clearly extends to tensor products of different degradable channels $\Phi_1 \otimes \Phi_2$ and hence implies $I^{\mathrm{coh}}(\Phi^{\otimes m}) = mI^{\mathrm{coh}}(\Phi)$.

A.3 Properties of antidegradable channels

In this section, we show that the set of antidegradable channels is convex. To do this, we first prove another result that is of independent interest.

Lemma 17 *Let $\Gamma : \mathcal{B}(\mathcal{H}_A) \mapsto \mathcal{B}(\mathcal{H}_B)$ be an anti-degradable CPT map and $\Delta : \mathcal{B}(\mathcal{H}_B) \mapsto \mathcal{B}(\mathcal{H}_C)$ any CPT map. Then the channel $\Delta \circ \Gamma$ is also anti-degradable.*

Proof: Let $\mathcal{H}_E$ and $\mathcal{H}_D$ be the environments for Γ and Δ respectively and let $U : \mathcal{H}_A \mapsto \mathcal{H}_{BE}$ and $V : \mathcal{H}_B \mapsto \mathcal{H}_{CD}$ denote the corresponding partial isometries for their Stinespring representations as in (1). Then the complement of $\Delta \circ \Gamma$ maps $\mathcal{B}(\mathcal{H}_A) \mapsto \mathcal{B}(\mathcal{H}_{DE})$ and satisfies

$$(\Delta \circ \Gamma)^C(\rho) = \mathrm{Tr}_C\,(V \otimes I_E) U \rho U^\dagger (V^\dagger \otimes I_E). \tag{73}$$

Furthermore, since the range of V is $\mathcal{H}_{CD}$ and $V^\dagger V = I_B$

$$\begin{aligned} \mathrm{Tr}_D\,(\Delta \circ \Gamma)^C(\rho) &= \mathrm{Tr}_{CD}\, U \rho U^\dagger (V^\dagger \otimes I_E)(V \otimes I_E) \\ &= \mathrm{Tr}_C\, U \rho U^\dagger \;=\; \Gamma^C(\rho). \end{aligned} \tag{74}$$

By assumption, there is a channel $\Lambda : \mathcal{H}_E \mapsto \mathcal{H}_B$ such that $\Lambda \circ \Gamma^C = \Gamma$. But then

$$(\Delta \circ \Lambda \circ \mathrm{Tr}_D)(\Delta \circ \Gamma)^C(\rho) = \Delta \circ \Gamma(\rho), \tag{75}$$

which implies that $\Delta \circ \Gamma$ is antidegradable. **QED**

Theorem 18 *The set of anti-degradable channels is convex.*

Proof: Let Φ_0 and Φ_1 be antidegradable channels and consider the channel

$$\Gamma = (1-p)\Phi_0 \otimes |0\rangle\langle 0|_F + p\Phi_1 \otimes |1\rangle\langle 1|_F, \tag{76}$$

whose complement is

$$\Gamma^C = (1-p)\Phi_0^C \otimes |0\rangle\langle 0|_G + p\Phi_1^C \otimes |1\rangle\langle 1|_G. \tag{77}$$

By assumption, there exist Ψ_j such that $\Psi_j \circ \Phi_j^C = \Phi_j$. With the Kraus operators of Ψ_j denoted $\{A_k^j\}_k$, define

$$A_j = A_k^0 \otimes |0\rangle\langle 0| + A_k^1 \otimes |1\rangle\langle 1|, \tag{78}$$

and let Ψ be the channel with Kraus operators A_k. Then

$$\begin{aligned} \Psi \circ \Gamma^C &= (1-p)(\Psi_0 \circ \Phi_0^C) \otimes |0\rangle\langle 0| + p(\Psi_1 \circ \Phi_1^C) \otimes |1\rangle\langle 1| \\ &= (1-p)\Phi_0 \circ \otimes |0\rangle\langle 0| + p\Phi_1 \circ \otimes |1\rangle\langle 1| = \Gamma \end{aligned} \tag{79}$$

so that Γ is antidegradable. Then applying Lemma 17 with $\Delta = \mathrm{Tr}_F$ implies that the channel

$$\mathrm{Tr}_F\,(1-p)\Phi_0 \otimes |0\rangle\langle 0|_F + p\Phi_1 \otimes |1\rangle\langle 1|_F) = (1-p)\Phi_0 + p\Phi_1. \tag{80}$$

is anti-degradable. This proves that the convex combination $\Phi = (1-p)\Phi_0 + p\Phi_1$ is anti-degradable. **QED**

B Qubit channels

B.1 Qubit channel representations and conditions

We first recall some well-known facts about qubit channels from [24] and [31]. A linear map $\Phi : M_2 \mapsto M_2$ can be represented by the matrix T_Φ with elements $\operatorname{Tr} \sigma_j \Phi(\sigma_k)$. When Φ has the form

$$\Phi : \tfrac{1}{2}[I + \sum_j w_k \sigma_k] \longmapsto \tfrac{1}{2}[I + \sum_k (t_k + \lambda_k w_k)\sigma_k)] \tag{81}$$

this matrix is

$$T_\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_1 & \lambda_1 & 0 & 0 \\ t_2 & 0 & \lambda_2 & 0 \\ t_3 & 0 & 0 & \lambda_3 \end{pmatrix}. \tag{82}$$

It was shown in [31] that when $t_1 = t_2 = 0$ a linear map of the form (81) is completely positive (CP) if and only if all $|\lambda_k| \leq 1$ and

$$(\lambda_1 \pm \lambda_2)^2 \leq (1 \pm \lambda_3)^2 - t_3^2, \tag{83}$$

and that the map has Choi rank $d_E \leq 2$, if and only if equality holds in (83). In that case,

$$\lambda_3 = \lambda_1 \lambda_2, \quad \text{and} \quad t_3^2 = (1 - \lambda_1^2)(1 - \lambda_2^2). \tag{84}$$

Channels satisfying (84) can can be represented by the matrix

$$T_\Phi = \begin{pmatrix} 1 & 0 & 0 & \\ 0 & \cos u & 0 & 0 \\ 0 & 0 & \cos v & 0 \\ \sin u \sin v & 0 & 0 & \cos u \cos v \end{pmatrix} \tag{85}$$

with $u = \cos^{-1}(\lambda_1), v = \cos^{-1}(\lambda_1)$.

As noted after Theorem 7, up to unitary conjugations, unital qubit maps can be written as $\Phi(\rho) = \sum_{k=0}^{3} a_k^2 \, \sigma_k \rho \, \sigma_k$ with $\sum_k a_k^2 = 1$ and the convention $\sigma_0 = I$. The matrix representative (82) has $t_i = 0$ and

$$\lambda_k = a_0^2 + a_k^2 - a_i^2 - a_j^2 \tag{86}$$

with i, j, k distinct, or, equivalently,

$$\begin{pmatrix} 1 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} a_0^2 \\ a_1^2 \\ a_2^2 \\ a_3^2 \end{pmatrix}. \tag{87}$$

B.2 Proof of Theorem 5

We now present a proof of Theorem 5 different from that in [40]. Denote a channel parameterized as in (85) by $\Phi(u,v)$. The amplitude-damping channels are those with $u=v$ and satisfy $\Phi(u_1,u_1)\circ\Phi(u_2,u_2)=\Phi(u_3,u_3)$ with $u_3=\cos^{-1}(\cos u_1\cos u_2)$. However, it is *not* true in general that $\Phi(u_1,v_1)\circ\Phi(u_2,v_2)=\Phi(u_3,v_3)$. The next theorem shows that this holds in a very special case.

Theorem 19 *Let $\Phi(u,v)$ be a qubit channel of the form* (85). *Then*

a) $\Phi^C(u,v)=\Phi(v-\frac{\pi}{2},u-\frac{\pi}{2})$, *and*

b) if $|\sin v|\le|\cos u|$, $\Phi(\theta_1,\theta_2)\circ\Phi(u,v)=\Phi(v-\frac{\pi}{2},u-\frac{\pi}{2})$ *with*

$$\theta_1=\cos^{-1}(\sin v/\cos u)\qquad\theta_2=\cos^{-1}(\sin u/\cos v). \tag{88}$$

Combining Theorem 19 with the fact that the Kraus operators for (85) are A_+ and A_- defined in (39) yields Theorem 5. Note that part (a) implies that a simple algorithm to map $\Phi\leftrightarrow\Phi^C$ is to change $\cos u\leftrightarrow\sin v$ and $\cos v\leftrightarrow\sin u$. (It is important that one change *both* $\sin\leftrightarrow\cos$ *and* $u\leftrightarrow v$.)

Proof of Theorem 19: To prove (a), we begin with the fact [31] that the Kraus operators for (85) are A_+ and A_- defined in (39) which we write in the in the compact form

$$F_1=A_+=aI+b\sigma_z\qquad F_2=A_-=c\sigma_x+id\sigma_y.$$

Next, we use the observation [23, Eq. (6)] that if $\Phi(\rho)=\sum_k F_k\rho F_k^\dagger$ then $\Phi^C(\rho)$ is the matrix with elements $\operatorname{Tr}F_j\rho F_k^\dagger$. Then for $\rho=\frac{1}{2}\big[I+\sum_j w_j\sigma_j\big]$ a straightforward computation gives

$$\begin{aligned}\Phi^C(\rho)&=\begin{pmatrix}a^2+b^2+2ab\,w_3 & (ac+bd)w_1-i(bc+ad)\,w_2\\ (ac+bd)w_1+i(bc+ad)\,w_2 & c^2+d^2-2ab\,w_3\end{pmatrix}\\&=\tfrac{1}{2}\big[I+\sin v\,w_1\sigma_1+\sin u\,w_2\sigma_2+(\cos u\cos v+\sin u\sin v\,w_3)\sigma_3\big],\end{aligned}\tag{89}$$

which establishes part (a).

To prove part (b), we rewrite this in the form (85) and compute $\Phi^C\circ\Phi^{-1}$ to get

$$\begin{aligned}\Psi&=\begin{pmatrix}1&0&0&0\\0&\sin v&0&0\\0&0&\sin u&0\\ \cos u\cos v&0&0&\sin u\sin v\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&\frac{1}{\cos u}&0&0\\0&0&\frac{1}{\cos v}&0\\-\frac{sinu\sin v}{\cos u\cos v}&0&0&\frac{1}{\cos u\cos v}\end{pmatrix}\\&=\begin{pmatrix}1&0&0&0\\0&\frac{\sin v}{\cos u}&0&0\\0&0&\frac{\sin u}{\cos v}&0\\ \cos u\cos v-\frac{sin^2u\sin^2 v}{\cos u\cos v}&0&0&\frac{sinu\sin v}{\cos u\cos v}\end{pmatrix}.\end{aligned}\tag{90}$$

To see if Ψ is CPT, we first apply the necessary condition $|\lambda_j| \leq 1$ for $j = 1, 2, 3$ to (90). Since $|\sin v| \leq |\cos u| \Leftrightarrow |\sin u| \leq |\cos v|$ this condition is either satsified for all λ_j or for none (if it is none, the map will be anti-degradable.) Then it suffices to see if (83) holds.

$$\begin{aligned}
t_3^2 &= \frac{1}{\cos^2 u \, \cos^2 v}\big(\cos^2 u \, \cos^2 v - \sin^2 u \, \sin^2 v\big)^2 \\
&= \frac{1}{\cos^2 u \, \cos^2 v}\big(\cos^2 u \,(1 - \sin^2 v) - (1 - \cos^2 u)\sin^2 v\big)^2 \\
&= \frac{1}{\cos^2 u \, \cos^2 v}\big(\cos^2 u - \sin^2 v\big)^2 \\
&= \frac{1}{\cos^2 u \, \cos^2 v}(\cos^2 u - \sin^2 v)(\cos^2 v - \sin^2 u) \\
&= \big(1 - \frac{\sin^2 v}{\cos^2 u}\big)\big(1 - \frac{\sin^2 u}{\cos^2 v}\big) \;=\; (1 - \lambda_1^2)(1 - \lambda_2^2).
\end{aligned} \tag{91}$$

Thus, Ψ is not only CP, it is also a map of the form (85) with $\lambda_1 = \cos\theta_1 = \frac{\sin v}{\cos u}$ and $\lambda_2 = \cos\theta_2 = \frac{\sin u}{\cos v}$, or equivalently $\Phi(\theta_1, \theta_2)$ with θ_j given by (88). Thus (90) becomes $\Phi(\theta_1, \theta_2) = \Phi(v - \frac{\pi}{2}, u - \frac{\pi}{2}) \circ \Phi^{-1}(u, v)$ which implies part (b). **QED**

B.3 Anti-degradable unital qubit channels

Since the Kraus operators for a unital qubit channel are $a_k \sigma_k$, it follows from [23, Eq. (4.2)] that

$$\Phi^C(\rho) = \begin{pmatrix} a_0^2 & a_0 a_1 & a_0 a_2 & a_0 a_3 \\ a_1 a_0 & a_1^2 & a_1 a_2 & a_1 a_3 \\ a_2 a_0 & a_2 a_1 & a_2^2 & a_2 a_3 \\ a_3 a_0 & a_3 a_1 & a_3 a_2 & a_3^2 \end{pmatrix} * \begin{pmatrix} w_0 & w_1 & w_2 & w_3 \\ w_1 & w_0 & -iw_3 & iw_2 \\ w_2 & iw_3 & w_0 & -iw_1 \\ w_3 & -iw_2 & iw_1 & w_0 \end{pmatrix}, \tag{92}$$

where $*$ denotes the pointwise Hadamard product. Φ^C can be represented by the 16×4 matrix with elements $\mathrm{Tr}\, |e_j\rangle\langle e_k|\Phi^C(\sigma_m)$

$$\begin{pmatrix} a_0^2 & 0 & 0 & 0 \\ 0 & a_0 a_1 & 0 & 0 \\ 0 & 0 & a_0 a_2 & 0 \\ 0 & 0 & 0 & a_0 a_3 \\ \\ 0 & a_1 a_0 & 0 & 0 \\ a_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\, a_1 a_2 \\ 0 & 0 & i\, a_1 a_3 & 0 \\ \\ 0 & 0 & a_2 a_0 & 0 \\ 0 & 0 & 0 & i\, a_2 a_1 \\ a_2^2 & 0 & 0 & 0 \\ 0 & -i\, a_2 a_3 & 0 & 0 \\ \\ 0 & 0 & 0 & a_3 a_0 \\ 0 & 0 & -i\, a_3 a_1 & 0 \\ 0 & i\, a_3 a_2 & 0 & 0 \\ a_3^2 & 0 & 0 & 0 \end{pmatrix}. \tag{93}$$

If Φ is anti-degradable, i.e. $\Psi \circ \Phi^C = \Phi$, the map Ψ can be represented by a 4×16 matrix with elements $\mathrm{Tr}\, \sigma_n \Psi(|e_j\rangle\langle e_k|)$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \; 0 & 1 & 0 & 0 & \; 0 & 0 & 1 & 0 & \; 0 & 0 & 0 & 1 \\ 0 & x_1 & 0 & 0 & \; x_1 & 0 & 0 & 0 & \; 0 & 0 & 0 & iy_1 & \; 0 & 0 & -iy_1 & 0 \\ 0 & 0 & x_2 & 0 & \; 0 & 0 & 0 & -iy_2 & \; x_2 & 0 & 0 & 0 & \; 0 & iy_2 & 0 & 0 \\ 0 & 0 & 0 & x_3 & \; 0 & 0 & iy_3 & 0 & \; 0 & -iy_3 & 0 & 0 & \; x_3 & 0 & 0 & 0 \end{pmatrix} \tag{94}$$

where x_k and y_k must be chosen to satisfy

$$2a_0 a_k\, x_k + 2a_i a_j\, y_k = a_0^2 + a_k^2 - a_i^2 - a_j^2 = \lambda_k \tag{95}$$

with i, j, k distinct. Although there are many solutions for x_k, y_k, only those which yield a CP map are acceptable. To check this, one needs to find the Choi matrix of Ψ. Each column of (94) defines one of the blocks in the Pauli basis, e.g., the block in the row 1 and col 3 is $x_2\sigma_y$. Thus the full Choi matrix for Ψ is

$$\begin{pmatrix} 1 & 0 & 0 & x_1 & 0 & -ix_2 & x_3 & 0 \\ 0 & 1 & x_1 & 0 & ix_2 & 0 & 0 & -x_3 \\ & & & & & & & \\ 0 & x_1 & 1 & 0 & iy_3 & 0 & 0 & -y_2 \\ x_1 & 0 & 0 & 1 & 0 & -iy_3 & y_2 & 0 \\ & & & & & & & \\ 0 & -ix_2 & -iy_3 & 0 & 1 & 0 & 0 & iy_1 \\ ix_2 & 0 & 0 & iy_3 & 0 & 1 & iy_1 & 0 \\ & & & & & & & \\ x_3 & 0 & 0 & y_2 & 0 & -iy_1 & 1 & 0 \\ 0 & -x_3 & -y_2 & 0 & -iy_1 & 0 & 0 & 1 \end{pmatrix}. \tag{96}$$

By conjugating with a suitable permutaton matrix, one can see that this contains two blocks, both unitarily equivalent to the matrices

$$\widetilde{Y} = \begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ x_1 & 1 & y_3 & y_2 \\ x_2 & y_3 & 1 & y_1 \\ x_3 & y_2 & y_1 & 1 \end{pmatrix} \qquad Y = \begin{pmatrix} 1 & x_3 & x_1 & x_2 \\ x_3 & 1 & y_2 & y_1 \\ x_1 & y_2 & 1 & y_3 \\ x_2 & y_1 & y_3 & 1 \end{pmatrix}. \tag{97}$$

The matrix $\widetilde{Y}$ is, up to phase factors, embedded in (96); an additional permutation yields the unitarily equivalent matrix Y, which we prefer to use. Thus, (96) is positive semi-definite if and only if (97) is, which requires

$$|x_k| \leq 1 \qquad \text{and} \qquad |y_k| \leq 1. \tag{98}$$

When (98) holds, (97) is positive semi-defininite if and only if

$$\begin{pmatrix} x_1 & x_2 \\ y_2 & y_1 \end{pmatrix} \begin{pmatrix} 1 & y_3 \\ y_3 & 1 \end{pmatrix}^{-1} \begin{pmatrix} x_1 & y_2 \\ x_2 & y_1 \end{pmatrix} \leq \begin{pmatrix} 1 & x_3 \\ x_3 & 1 \end{pmatrix}. \tag{99}$$

Using $\left(\begin{smallmatrix} 1 & y \\ y & 1 \end{smallmatrix}\right)^{-1} = \frac{1}{1-y^2}\left(\begin{smallmatrix} 1 & -y \\ -y & 1 \end{smallmatrix}\right)$ this is straightforward to evaluate. In some cases, conjugating with the Hadamard gate $H = 2^{-1/2}\left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$ gives a more useful expression. In particular,

- When $x_1 = x_3$ and $y_1 = y_3$, (99) becomes

  $$\tfrac{1}{1+y_3} \begin{pmatrix} (x_1+y_1)^2 & x_1^2 - y_1^2 \\ x_1^2 - y_1^2 & (x_1-y_1)^2 \end{pmatrix} \leq \begin{pmatrix} 1+x_3 & 0 \\ 0 & 1-x_3 \end{pmatrix} \tag{100}$$

  or, equivalently,

  $$(1 + y_3 - x_1^2 - y_1^2)I - (x_1^2 - y_1^2)\sigma_3 + [x_3(1+y_3) + 2x_1y_1]\sigma_3 \geq 0. \tag{101}$$

- When $x_k = y_k$, the matrix$(H \otimes I)Y(H \otimes I)$ is precisely the Choi matrix of the unital map

$$\tfrac{1}{2}[I + \sum_j x_k \sigma_k] \longmapsto \tfrac{1}{2}[I + \sum_k \lambda_k x_k \sigma_k]. \tag{102}$$

  Thus, Ψ is CP if and only if $|x_k| \leq 1$ and $(x_1 \pm x_2)^2 \leq (1 \pm x_3)^2$. This can also be seen by obvserving that conjugating both sides of (99) with H yields diagonal matrices satisfying

$$(x_1 I + x_2 \sigma_3)(I + x_3 \sigma_3)^{-1}(x_1 I + x_2 \sigma_3) \leq (I + x_3 \sigma_3).$$

It should be pointed out that (94) is not the most general possible degrading map. For example, one could change its first row to

$$\begin{pmatrix} 1 & t & 0 & 0 & \quad -t & 1 & 0 & 0 & \quad 0 & 0 & 1 & 0 & \quad 0 & 0 & 0 & 1 \end{pmatrix}$$

This will not affect (95), but it will introduce non-zero cross-terms in the block structure used to reduce the positivity of (96) to that of (99). The positivity of (99) will still be necessary, but the cross-terms will introduce additional constraints without relaxing any other requirements. Thus, there is no loss of generality in assuming that the degrading map has the form Ψ.

B.4 Anti-degradable channels with one $a_k = 0$

For notational simplicity, we assume $a_0 = 0$ and $a_j \neq 0$ for $j = 1, 2, 3$. Then we can assume $x_k = 0$ and, with i, j, k distinct

$$y_k = \frac{2a_k^2 - 1}{2a_i a_j} = \frac{1 - 2a_i^2 - 2a_j^2}{2a_i a_j} \tag{103}$$

When $x_k = 0$, (99) becomes

$$y_1^2 + y_2^2 + y_3^2 - 2y_1 y_2 y_3 \leq 1 \tag{104}$$

which is equivalent to the condition that the 3×3 subdeterminant of (97) is ≥ 0.

When $a_1^2 = a_2^2, a_3^2 - 1 = 2a_1^2$, and $y_1 = y_2 = \frac{1a^2 - 1}{2a\sqrt{1 - 2a^2}}, y_3 = \frac{1 - 4a^2}{2a^2}$, the condition (104) is equivalent to $2y_1^2(1 - y_3) \leq 1 - y_3^2$ so that for $y_3 \neq \pm 1$, (104) is equivalent to $2y_1^2 \leq 1 + y_3$. But since $2y_1^2 = \frac{1 - 2a^2}{2a^2} = 1 + y_3$, (104) always holds with equality when we choose $y_1 = y_2$. Moreover, the choice, $y_1 = -y_2$ gives a stronger condition when $y_3 > 0$, but does not yield additional solutions.

In the general case $a_1 \neq a_2$, substituting (103) into (104) gives

$$4a_1^2 a_2^2 a_3^2 \geq a_1^2(2a_1^2 - 1)^2 + a_2^2(2a_2^2 - 1)^2 + a_3^2(2a_3^2 - 1)^2 - (2a_1^2 - 1)(2a_2^2 - 1)(2a_3^2 - 1) \tag{105}$$

By using $a_k^2 = 1 - a_i^2 - a_j^2$ this can be reduced to an inequality in two variables, which, perhaps surprisingly, can also be shown to hold with equality after some rather tedious algebra.Thus, in the situation considered here with the choices above for y_k, the matrix (99) is positive semi-definite if all $y_k^2 \leq 1$.

The condition $|y_3|^2 \leq 1$ becomes

$$(1 - 2a_1^2 - 2a_2^2)^2 \leq 4a_1^2 a_2^2 \tag{106}$$

After inverting (87) and substituting, one finds

$$4\lambda_3^2 \leq (1 - \lambda_3)^2 - (\lambda_1 - \lambda_2)^2 \tag{107}$$

or, equivalently,

$$(\lambda_1 - \lambda_2)^2 \leq (1 - 3\lambda_3)(1 + \lambda_3) \tag{108}$$

which implies $-1 \leq \lambda_3 \leq \frac{1}{3}$. The conditions for $k = 1, 2$ are equivalent. Thus, a necessary and sufficient condition that a channel with $a_0 = 0$ is anti-degradable is

$$(\lambda_i - \lambda_j)^2 \leq (1 - 3\lambda_k)(1 + \lambda_k) \tag{109}$$

for any permutation $\{i, j, k\}$ of $\{1, 2, 3\}$. Similar conditions hold if $a_n^2 = 0$ for $n = 1, 2, 3$ and a_n replaced by a_0.

Recall that in a fixed basis, the unital qubit maps correspond to a tetrahedron with the multiplier $[\lambda_1, \lambda_2, \lambda_3]$ defining a point in in $\mathbf{R}_3$. Each condition $a_n^2 = 0$ describes a triangular "face" of this tetrahedron. In particular, the face with $a_0^2 = 0$ is the convex hull of the 3 points

$$[1, -1, -1], \quad [-1, +1, -1], \quad [-1, -1, 1]$$

corresponding to conjugation with σ_k for $k = 1, 2, 3$ respectively. See Fig. 1.

- Each of the edges of the tetrahedron and, hence, the edges of the face correspond to degradable channels, with only the midpoints anti-degradable as well as degradable.

- The EB maps correspond to triangles whose vertices are midpoints of the edges of the face, i.e., maps whose multipliers are permutations of $[0, 0, \pm 1]$.

- The boundary of the anti-degradable region is described by curves obtained as the intersection of the surface of points for which equality holds in (109) with a face. Projected onto one of the faces, these curves form a circle.

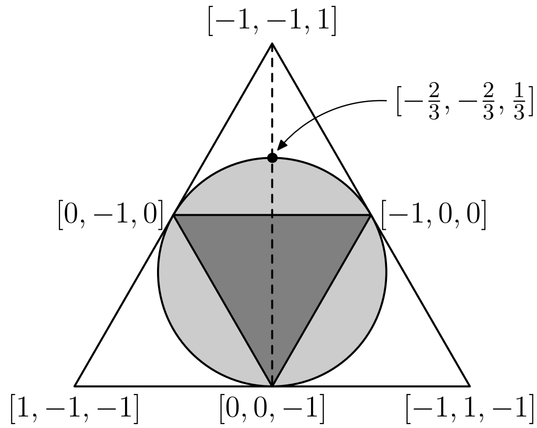

Figure 1: The face of the tetrahedron of unital qubit maps with $a_0^2 = 0$. The EB region is the small darkly shaded triangle; the anti-degradable region is the circle and its interior. The dashed line corresponds to channels with multiplier $[-x,-x,2x-1]$ unitarily equivalent to two-Pauli channels, with the extreme anti-degradable one marked with a dot.

The so-called "two-Pauli" channel has (up to permutations of $1,2,3$), $a_3 = 0$, $a_1 = a_2 = t, a_0 = \sqrt{1-2t^2}$ with $0 < t^2 \leq \frac{1}{2}$ and multiplier $[1-2t^2, 1-2t^2, 1-4t^2]$. Switching $a_0 \leftrightarrow a_3$ does not change the analysis above in any essential way; it suffices to set $y_3 = 0$ and replace y_3 by x_3 in (103) and what follows. Moreover, this change does not affect (106) which becomes $|1-4t^2| \leq 2t^2$. Thus, we can conclude that a two-Paul channel is anti-degradable if and only if $\frac{1}{6} \leq t^2 \leq \frac{1}{2}$. This is larger than the entanglement breaking range $\frac{1}{4} \leq t^2 \leq \frac{1}{2}$, and thus gives (after including permutations and conjugations) 12 new extreme points of the anti-degradable channels with $t^2 = \frac{1}{6}$, e.g., corresponding to multiplier $[\frac{2}{3}, \frac{2}{3}, \frac{1}{3}]$. Conjugating this with σ_3 gives a family of channels with multipliers of the form $[-x,-x,2x-1]$ with $0 \leq x = 1-2t^2 \leq 1$ corresponding to the dashed line shown in Figure 1.

B.5 Anti-degradable depolarizing channel

For the depolarizing channel with $a_k = a$ for $k \neq 0$ and $a_0 = \sqrt{1-3a^2}$ and the assumption of symmetric solutions $x_k = x$, $y_k = y$, (95) becomes

$$2a\sqrt{1-3a^2}x + 2a^2 y = 1 - 4a^2. \tag{110}$$

When $a^2 = \frac{1}{12}$, (110) becomes $\frac{1}{2}x + \frac{1}{6}y$ whose only solution in the unit square is $x = y = 1$, for which (97) is a multiple of a rank one projection and hence, on the boundary of the cone of positive semi-definite matrices. In the entanglement-breaking region $\frac{1}{6} \leq a^2 \leq \frac{1}{3}$, $x = 0, y = \frac{1-4a^2}{2a^2}$ always gives a solution for which (97) is positive semi-definite. For the general case, observe that when $x_1 = x_2 = x$ and $y_1 = y_2 = y$, (101) holds if and only if $1 + y \geq x^2 + y^2$ and $(1 + y - x^2 - y^2)^2 \geq (x^2 - y^2)^2 + x^2(1 - y)^2$. The latter inequality is stronger in the unit square, and can be rewritten as

$$(1 + y)^2 - 2(x^2 + y^2) - 2y^3 + 3x^2y^2 - x^2 \geq 0. \tag{111}$$

Then for $\frac{1}{12} < a^2 < \frac{1}{3}$ one has a family of **non-unique** solutions corresponding to the line segment which satisfies (110) and lies within the region in the xy-plane bounded above by the line $y = 1$ and below by curve for which equality holds in (111), as shown in Figure 2. Thus, we have recovered the well-known result [4, 6] that depolarizing channels with $|\lambda_k| \leq \frac{2}{3}$ are anti-degradable. Moreover, we have shown that, except for $\lambda_k = \frac{2}{3}$ and $\lambda_k = -\frac{1}{3}$, the degrading map for Φ^C is *not* unique.

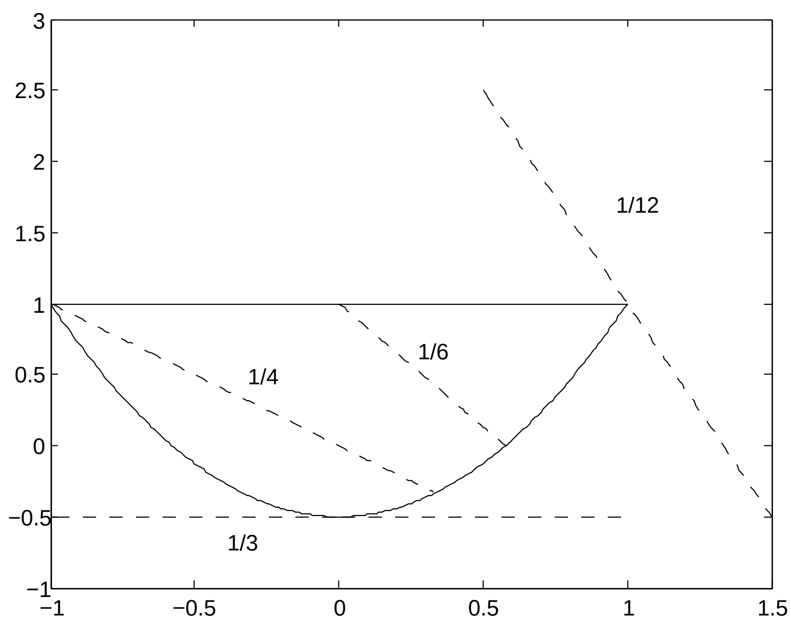

Figure 2: The solution region for the depolarizing channel satisfying (111). For $a^2 = \frac{1}{12}$ the line (110) is $3x + y = 1$ which yields the unique solution $x = y = 1$. At the EB boundary $a^2 = \frac{1}{6}$ the solutions lie on the line from $(0, 1)$ to $(\frac{1}{\sqrt{3}}, 0)$; and at $a^2 = \frac{1}{4}$ on the line from $(-1, 1)$ to $(\frac{1}{3}, -\frac{1}{3})$. At $a^2 = \frac{1}{3}$ one has only the unique solution $(0, -0.5)$.

B.6 Proof of Theorems 7 and 8

To study the general case of unital qubit channels, first consider the situation in which all $\lambda_k \geq 0$ and all $a_j \geq 0$. We will then show that the latter does not involve any loss of generality and that channels with some $\lambda_k \leq 0$ situations are either entanglement breaking or can be rotated into the positive case by conjugating with a σ_j.

First, observe that all $\lambda_k > 0$ implies

$$0 < \lambda_i + \lambda_j = 2(a_0^2 - a_k^2) \tag{112}$$

with $\{i, j, k\}$ any permutation of $\{1, 2, 3\}$. Therefore, $a_0^2 > a_k^2$ for $k = 1, 2, 3$. Combing this with our assumption that all $a_k \geq 0$, we can conclude that $a_0 \geq a_k$ for $k = 1, 2, 3$.

Next observe that the requirement that x_k, y_k lie in the unit square, implies that the absolute value of the LHS of (95) is bounded above by $2a_0 a_k + 2a_i a_j$. Thus, a necessary condition for anti-degradability is that

$$\lambda_k = a_0^2 + a_k^2 - a_i^2 - a_j^2 \leq 2a_0 a_k + 2a_i a_j \tag{113}$$

which is equivalent to $(a_0 - a_k)^2 \leq (a_i + a_j)^2$. With the assummption that all $a_j \geq 0$, this implies

$$a_0 \leq a_i + a_j + a_k. \tag{114}$$

Substituting (114) into (113) and using $a_0^2 = 1 - a_i^2 - a_j^2 - a_k^2$ gives (40) as a necessary condition for antidegradability in the case $\{i, j, k\} = \{1, 2, 3\}$. In the multiplier picture this becomes (still assuming all $a_j \geq 0$)

$$\sum_{k=1}^{3} \left(1 - \lambda_k + \sqrt{(1 - \lambda_k)^2 - (\lambda_i - \lambda_j)^2}\right) \geq 2 \tag{115}$$

with all i, j, k distinct.

To show that (115) is sufficient for anti-degradability, it is enough to verify that $x_k = y_k = \frac{a_0^2 + a_k^2 - a_i^2 - a_j^2}{2a_0 a_k + 2a_i a_j}$ yields a CPT degrading map for a Pauli channel with multipliers $\lambda_k = a_0^2 + a_k^2 - a_i^2 - a_j^2$. When all $\lambda_k \geq 0$, and $a_k \geq 0$, the condition $0 \leq x_k = y_k \leq 1$ is equivalent to $(a_0 - a_k)^2 \leq (a_i + a_j)^2$ which is equivalent to (114). Since (115) is equivalent to (114) when all $a_k > 0$, we have shown that it is also sufficient for degradability.

Now a unital qubit channel is independent of the choice of phase for the Kraus operators $a_k \sigma_k$. Hence, its degradability can not depend on this phase either, although allowing non-poisitive a_k might yield additional degrading maps. Thus, (115) is necessary and sufficient for degradability when all $\lambda_k > 0$. The corresponding surface in this quadrant is shown in Figure 3.

To complete the proof of Theorem 7 it suffices to observe that conjugating with σ_k replaces $1, 2, 3$ in (40) by $0, i, j$ with i, j, k distinct in $\{1, 2, 3\}$. The corresponding version of (115), has signs modified so that $\lambda_j \mapsto -\lambda_j$ for $j \neq k$, and (115) becomes (42)

$$\sum_{k=1}^{3} \Big(1 - |\lambda_k| + \sqrt{(1-|\lambda_k|)^2 - (|\lambda_i| - |\lambda_j|)^2}\Big) \geq 2.$$

Note that the CP condition (83) with $t_j = 0$ implies that the quantities under the square root in (115) are non-negative. This remains true in (42) because changing the sign of two λ_j either leaves $(1-\lambda_k)^2 - (\lambda_i - \lambda_j)^2$ unchanged or changes it to $(1+\lambda_k)^2 - (\lambda_i + \lambda_j)^2$, which is also non-negative by (83). One way of charactering the unital EB class [32] is that (83) is replaced by the stronger conditions

$$(1 \pm \lambda_k)^2 - (\lambda_i \mp \lambda_j)^2 \geq 0, \tag{116}$$

which is equivalent to $\sum_k |\lambda_k| \leq 1$ and immediately implies (42).

Another way of viewing this situation is to observe that interior of the well-known tetrahedron of unital qubit maps can be written as the union of 8 regions:

- 4 hexahedrons with an even number of λ_n negative. One corresponds to all $\lambda_n > 0$; the others can be obtained from this by conjugating with $\sigma_k,\ k = 1,2,3$ for which $\lambda_k > 0$ and the remaining two $\lambda_j < 0$.
- 4 tetrahedrons with an odd number of λ_n negative. One corresponds to all $\lambda_n < 0$; the others can be obtained from this by conjugating with $\sigma_k,\ k = 1,2,3$ for which $\lambda_k < 0$ and the remaining two $\lambda_j > 0$.

It was shown in [32] that any channel which remains CP when $\lambda_k \mapsto -\lambda_k$ (which is equivalent to applying the partial transpose to the Choi matrix and conjugating with a Puali matrix) is EB. It follows that all unital qubit channels with an odd number of λ_n negative, or any $\lambda_n = 0$, is EB. Moreover, a unital qubit channel is EB if and only if $\sum_k |\lambda_k| \leq 1$ which implies that (42). Thus we have proven Theorem 8. In the case of channels with an odd number of negative λ_k it can happen that a linear qubit map of the form (41) satisfies (42) without being CP. Therefore it is important that the CP condition $(1 \pm \lambda_k)^2 \geq (\lambda_i \pm \lambda_j)^2$ is included in the hypothesis.

Indeed, the astute reader will note that the proof found it sufficient to consider degrading maps Ψ with $x_k = y_k$. However, the constraints on the degrading map for depolarizing channels in Section B.5 imply that for $a^2 \approx \frac{1}{3}$ no solution with $x = \pm y$ exists. There is no contradiction because for $a^2 > \frac{1}{4}$, the multiplier $\lambda < 0$ and the assumption that a_0^2 is largest no longer holds. This does, however, demonstrate the need to consider the four small tetrahedrons with an odd number of λ_n negative separately.

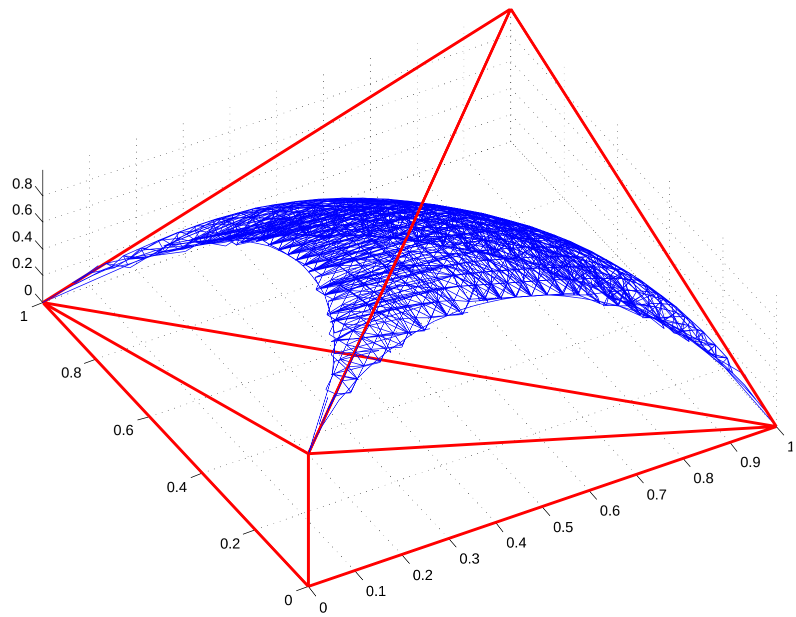

Figure 3: The hexahedron of unital qubit maps in the sector with all $\lambda_k \geq 0$, also showing the boundary of the anti-degradable region. The tetrahedron on the bottom corresponds to the subset of EB channels.

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