

QMA-hardness of Local Hamiltonian

Recall:

Def (QMA + amplification)

Decision (YES/NO) problem, problem size n .

Problem \in QMA if

$\exists T = O(\text{poly}(n))$, $\epsilon = \Omega\left(\frac{1}{\text{poly}(n)}\right)$,
quantum circuit $U = U_T \cdot U_{T-1} \cdots U_2 \cdot U_1$ s.t.

YES:

$\exists |w\rangle: \Pr(U|w\rangle \text{ outputs "1"}) \geq 1 - \epsilon$

NO:

$\forall |w\rangle: \Pr(U|w\rangle \text{ outputs "1"}) \leq \epsilon$

Def (Local Hamiltonian Problem)

Input: k -local Hamiltonian H on
 n qudits with m local terms,
where $\lambda_0(H) \leq \alpha$ or $\lambda_0(H) \geq \beta$
with $\beta - \alpha \geq \frac{1}{\text{poly}(n)}$

Output: YES if $\lambda_0(H) \leq \alpha$
NO if $\lambda_0(H) \geq \beta$

Thm (Kitaev)

The Local Hamiltonian problem is
QMA-hard

We saw that entanglement defeats naive proof approach.

→ Need smarter way of encoding witness verification computation into Hamiltonian.

Idea (Feynman): encode evolution of computation in quantum superposition:

"Computational history state" or "history state":

$$|\Psi\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^T |\psi_t\rangle |t\rangle_c$$

computation register $(\mathbb{C}^2)^{\otimes n}$ clock register \mathbb{C}^{T+1}

$|\psi_t\rangle$ = state after t steps (gates)

$|\psi_0\rangle$ = initial input state

$|\psi_T\rangle$ = final output state

Clock register takes states $\in \mathbb{C}^{T+1}$:
 $|0\rangle, |1\rangle, |2\rangle, \dots, |t\rangle, \dots, |T-1\rangle, |T\rangle$

Easy to write down (non-local) Hamiltonian with $|\Psi\rangle$ as ground state:

$$H = H_{in} + H_{prop}$$

$$= (\mathbb{1} - |\psi_0\rangle\langle\psi_0|) \otimes |0\rangle\langle 0|$$

ensures correct initial state

$$+ \sum_{t=0}^{T-1} (|\psi_t\rangle\langle\psi_t| \otimes |t\rangle\langle t| + |\psi_{t+1}\rangle\langle\psi_{t+1}| \otimes |t+1\rangle\langle t+1|$$

forces clock to "tick"

$$- |\psi_{t+1}\rangle\langle\psi_t| \otimes |t+1\rangle\langle t|$$

forces 1 step of computation & increments clock

$$- |\psi_t\rangle\langle\psi_{t+1}| \otimes |t\rangle\langle t+1|$$

necessary for Hermiticity; forces 1 step of computation backwards in time & decrements clock.

Exercise 4

Show that unique g.s. of H is $|\Psi\rangle$.

Challenge is to:

- construct local H for QMA verifier
- prove YES instance $\Rightarrow \lambda_0(H) \leq \alpha$
- prove NO instance $\Rightarrow \lambda_0(H) \geq \beta$

$$\left(\beta - \alpha \geq \frac{1}{\text{poly}(n)} \quad \text{promise gap} \right)$$

Proof (Kitaev Thm.)

Hamiltonian (not fully local yet!)

$$H = H_{in} + H_{prop} + H_{out}$$

Initially, take

$$\mathcal{H} = (\mathbb{C}^2)^{\otimes n} \otimes \mathbb{C}^{T+1} = \underbrace{\mathbb{C}^2}_{\text{computation register}} \otimes \underbrace{(\mathbb{C}^2)^{\otimes |W|}}_{\substack{\text{output} \\ \text{qubit}}} \otimes \underbrace{(\mathbb{C}^2)^{\otimes |A|}}_{\substack{\text{witness} \\ \text{register} \\ \text{qubits } W}} \otimes \underbrace{\mathbb{C}^{T+1}}_{\text{ancillas } A}$$

computation register

(will make clock local later).

$$H_{in} := \Pi_1^{(1)} \otimes |0\rangle\langle 0|_{cl} + \sum_{j \in A} \Pi_j^{(1)} \otimes |0\rangle\langle 0|_{cl}$$

forces output qubit & ancillas to "initially" be in $|0\rangle$ state

$$H_{prop} := \sum_{t=1}^{T-1} H_t, \text{ where}$$

$$H_t := \frac{1}{2} \mathbb{1} \otimes (|t\rangle\langle t| + |t+1\rangle\langle t+1|)$$

forces clock to "tick"

$$- \frac{1}{2} U_{t+1} \otimes |t+1\rangle\langle t| - \frac{1}{2} U_{t+1}^\dagger \otimes |t\rangle\langle t+1|$$

forward computation step backwards step

$$H_{out} := \Pi_1^{(0)} \otimes |T\rangle\langle T|$$

gives energy "penalty" if output of QMA verifier computation is "0".

$$\begin{aligned}
&= \frac{1}{2} \left(\underbrace{(u_1^\dagger \dots u_t^\dagger)}_{=1} \underbrace{(u_t \dots u_1)}_{=1} \otimes |t \times t| \rangle \right. \\
&\quad + \underbrace{(u_1^\dagger \dots u_t^\dagger u_{t+1}^\dagger)}_{=1} \underbrace{(u_{t+1} u_t \dots u_1)}_{=1} \otimes |t+1 \times t+1| \\
&\quad - \underbrace{(u_1^\dagger \dots u_t^\dagger u_{t+1}^\dagger)}_{=1} u_{t+1} \underbrace{(u_t \dots u_1)}_{=1} \otimes |t+1 \times t| \\
&\quad \left. - \underbrace{(u_1^\dagger \dots u_t^\dagger)}_{=1} u_{t+1}^\dagger \underbrace{(u_{t+1} u_t \dots u_1)}_{=1} \otimes |t \times t+1| \right)
\end{aligned}$$

$$= \mathbb{1} \otimes |\phi_t \times \phi_t| \quad \text{where } |\phi_t\rangle = \frac{1}{\sqrt{2}} (|t\rangle - |t+1\rangle)$$

$$H_{\text{prop}} \sim W H_{\text{prop}} W^\dagger = \sum_t \mathbb{1} \otimes |\phi_t \times \phi_t| = \mathbb{1} \otimes E.$$

□

YES instance:

\exists witness $|w\rangle$ s.t.

$\Pr(\text{circuit outputs "0" on input } |w\rangle) \leq \epsilon$ (Def. QMA)

$$\text{Let } |\Psi\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^T |\psi_t\rangle |t\rangle_{cl}$$

$$\text{where } |\psi_t\rangle = U_t \cdot U_{t-1} \cdots U_2 \cdot U_1 |\psi_0\rangle$$

$$|\psi_0\rangle = |0\rangle |w\rangle \underbrace{|0\rangle \cdots |0\rangle}_{\text{ancillas}}$$

output qubit witness

Note:

$$\begin{aligned} \bullet \quad |\Psi\rangle &= \left(\sum_t U_t \cdots U_1 \otimes |t\rangle\langle t+1| \right) (|\psi_0\rangle \otimes \underbrace{\frac{1}{\sqrt{T+1}} \sum_t |t\rangle}_{|\varphi_0\rangle}) \\ &= W^\dagger |\psi_0\rangle \otimes |\varphi_0\rangle. \end{aligned}$$

$$\begin{aligned} \bullet \quad \Pr(\text{output "0" on input } |w\rangle) &= \langle \psi_T | \Pi_1^{(0)} | \psi_T \rangle \leq \epsilon. \end{aligned}$$

Want to show $\lambda_0(H) \leq \alpha$, α small
→ show computational history state has low energy.

$$\lambda_0(H) = \min_{|\phi\rangle} \langle \phi | H | \phi \rangle$$

$$\leq \langle \Psi | H | \Psi \rangle$$

$$= \langle \Psi | H_{in} | \Psi \rangle + \langle \Psi | H_{prop} | \Psi \rangle + \langle \Psi | H_{out} | \Psi \rangle$$

$$= \sim \sim + \sum_t \langle \Psi | H_t | \Psi \rangle + \sim \sim$$

$$\langle \Psi | H_{in} | \Psi \rangle$$

$$= \frac{1}{T+1} \left(\sum_{t'=0}^T \langle \psi_{t'} | \langle t' | \right) \left((\pi_1^{(1)} + \sum_{j \in \text{anc}} \pi_j^{(1)}) \otimes 10 \times_{cl} 01 \right) \cdot \left(\sum_{t=0}^T |\psi_t\rangle |t\rangle \right)$$

$$\propto \langle \psi_0 | \langle 0 | \left((\pi_1^{(1)} + \sum_{j \in \text{anc}} \pi_j^{(1)}) \otimes 10 \times_{cl} 01 \right) | \psi_0 \rangle | 0 \rangle$$

$$= \langle 0 | \langle w | \langle 0 |^{\otimes A} \cdot \pi_1^{(1)} \cdot | 0 \rangle | w \rangle | 0 \rangle$$

$$= \langle 0 | \pi_1^{(1)} | 0 \rangle = 0.$$

$$\langle \Psi | H_{prop} | \Psi \rangle$$

$$= \langle \psi_0 | \otimes \langle \varphi_0 | W H_{prop} W^\dagger | \psi_0 \rangle \otimes | \varphi_0 \rangle$$

$$= \langle \psi_0 | \otimes \langle \varphi_0 | \mathbb{1} \otimes E | \psi_0 \rangle \otimes | \varphi_0 \rangle \quad (W H_{prop} W^\dagger = \mathbb{1} \otimes E)$$

$$= \langle \varphi_0 | E | \varphi_0 \rangle$$

$$\propto \langle \varphi_0 | \sum_t (|t\rangle - |t+1\rangle) (\langle t| - \langle t+1|) \sum_{t'} |t'\rangle$$

$$= 0.$$

$$\begin{aligned}
& \langle \Psi | H_{\text{out}} | \Psi \rangle \\
&= \frac{1}{T+1} \left(\sum_{t'=0}^T \langle \psi_{t'} | \langle t' | \right) (\pi_1^{(0)} \otimes |T\rangle\langle T|) \cdot \\
&\quad \cdot \left(\sum_{t=0}^T | \psi_t \rangle | t \rangle \right) \\
&= \frac{1}{T+1} \langle \psi_T | \pi_1^{(0)} | \psi_T \rangle \\
&= \frac{1}{T+1} \text{Pr}(\text{output "0" on input } |\omega\rangle) \\
&\quad \text{(from above)} \\
&\leq \frac{\varepsilon}{T+1}
\end{aligned}$$

Thus

$$\begin{aligned}
\lambda_0(H) &\leq \langle \Psi | \overset{=0}{\cancel{H_{\text{in}}}} | \Psi \rangle + \sum_{t=1}^T \langle \Psi | \overset{=0}{\cancel{H_t}} | \Psi \rangle \\
&\quad + \langle \Psi | H_{\text{out}} | \Psi \rangle \\
&\leq \frac{\varepsilon}{T+1} \text{ YES instances .}
\end{aligned}$$

NO instance:

$\forall |w\rangle:$

$\Pr(\text{circuit outputs "0" on input } |w\rangle)$

$$= \langle \Psi_T | \Pi_1^{(0)} | \Psi_T \rangle$$

$$\geq 1 - \varepsilon \quad (\text{Def. QMA})$$

Need following:

Def (angle between subspaces)

Subspaces $\mathcal{L}_1, \mathcal{L}_2$.

$$\cos \theta(\mathcal{L}_1, \mathcal{L}_2) := \max_{\substack{|\varphi_1\rangle \in \mathcal{L}_1 \\ |\varphi_2\rangle \in \mathcal{L}_2}} |\langle \varphi_1 | \varphi_2 \rangle|$$

(cf. $\vec{u} \cdot \vec{v} = \cos \theta$, $|\vec{u}| = |\vec{v}| = 1$)

Lemma (Kitaev)

If $A, B \geq 0$ and

$\lambda_{\min}(A|_{\text{supp } A}), \lambda_{\min}(B|_{\text{supp } B}) \geq \mu$, then

$$A + B \geq 2\mu \sin^2 \frac{\theta}{2}$$



where $\theta = \theta(\ker A, \ker B)$.

(Recall $A \geq c \Leftrightarrow A - c\mathbb{1} \geq 0 \Leftrightarrow \lambda_{\min}(A) \geq c$.)

Proof

If $\ker A \cap \ker B \neq \emptyset$, $\Theta = 0$ & bound is trivially true. So assume wlog $\ker A \cap \ker B = \emptyset$.

Let $\pi_A =$ projector onto $\ker A$

$\pi_B =$  "  $\ker B$

$$A \geq \mu (\mathbb{1} - \pi_A), \quad B \geq \mu (\mathbb{1} - \pi_B) \quad (\text{trivial})$$

\rightarrow sufficient to prove

$$(\mathbb{1} - \pi_A) + (\mathbb{1} - \pi_B) \geq 2 \sin^2 \frac{\Theta}{2}$$

or

$$\pi_A + \pi_B \leq 1 + \cos \Theta. \quad (*)$$

$= 2 \cos^2 \frac{\Theta}{2}$

Let $|\psi\rangle$ be any eigenvect. of $\pi_A + \pi_B$ with eigenval λ (> 0 wlog; $(*)$ trivially true for $\lambda = 0$):

$$\lambda |\psi\rangle = (\pi_A + \pi_B) |\psi\rangle = a |\varphi_A\rangle + b |\varphi_B\rangle$$

where $\pi_A |\psi\rangle =: a |\varphi_A\rangle$, $\pi_B |\psi\rangle =: b |\varphi_B\rangle$

& wlog $a, b \in \mathbb{R}^+$ (absorb phases into $|\varphi_{A,B}\rangle$)

$$\lambda = \langle \psi | (\pi_A + \pi_B) | \psi \rangle$$

$$= \langle \psi | \pi_A \pi_A | \psi \rangle + \langle \psi | \pi_B \pi_B | \psi \rangle$$

($\pi^2 = \pi$ for projectors)

$$= a^2 + b^2$$

$$\begin{aligned}
\lambda^2 &= \langle \psi | (\pi_A + \pi_B)^2 | \psi \rangle \\
&= \underbrace{a^2 + b^2} + 2ab \operatorname{Re} \langle \varphi_A | \varphi_B \rangle \\
&\leq \lambda + 2ab |\operatorname{Re} \langle \varphi_A | \varphi_B \rangle| \quad a, b \in \mathbb{R}^+ \\
&\leq \lambda + (a^2 + b^2) |\operatorname{Re} \langle \varphi_A | \varphi_B \rangle| \\
&\quad \uparrow (a^2 + b^2 - 2ab = (a-b)^2 \geq 0 \Rightarrow 2ab \leq a^2 + b^2) \\
&= \lambda (1 + |\operatorname{Re} \langle \varphi_A | \varphi_B \rangle|) \\
&\leq \lambda \left(1 + \max_{\substack{|\varphi_A\rangle \in \ker A \\ |\varphi_B\rangle \in \ker B}} |\langle \varphi_A | \varphi_B \rangle| \right) \\
&= \lambda (1 + \cos \theta)
\end{aligned}$$

$\therefore \lambda \leq 1 + \cos \theta$ which proves (*). \square

Let $A = H_{in} + H_{out}$

$B = H_{prop} = \sum_t H_t$

in Lemma.

Need to bound $\lambda_{\min}(A|_{\text{supp}A})$, $\lambda_{\min}(B|_{\text{supp}B})$, $\Theta(\ker A, \ker B)$

Claim $\lambda_{\min}(A|_{\text{supp}A}) \geq 1$

smallest non-zero eigval

Proof: H_{in}, H_{out} commuting projectors

Claim $\lambda_{\min}(B|_{\text{supp}B}) \geq \Omega(T^{-2})$

Proof

Define unitary

$W := \sum_t (U_1^\dagger \cdot U_2^\dagger \cdots U_{t-1}^\dagger \cdot U_t^\dagger) \otimes |t\rangle\langle t|$

(We used this before in proof of $H \geq 0$.)

$B = H_{prop} \sim W H_{prop} W^\dagger = \mathbb{1} \otimes E$

where $E = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & & & & \\ -\frac{1}{2} & 1 & -\frac{1}{2} & & & \\ & -\frac{1}{2} & 1 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & & 1 & -\frac{1}{2} \\ & & & & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$

from before

\therefore Eigvals of $B =$ eigvals of E .

Eigenvals & eigenvects of E are:

$$\lambda_k = 1 - \cos q_k, \quad |\varphi_k\rangle \propto \sum_{j=0}^T \cos(q_k (j + \frac{1}{2})) |j\rangle$$

where

$$q_k = \frac{\pi k}{T+1}, \quad k = 0, \dots, T$$

Exercise Verify this.

$$\lambda_0 = 0, \quad |\varphi_0\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^T |t\rangle.$$

Smallest non-zero eigenval

$$\lambda_1 = 1 - \cos\left(\frac{\pi}{T+1}\right) \geq \Omega(T^{-2}). \quad \square$$

Claim $\sin^2 \frac{\theta}{2} = \Omega\left(\frac{1 - \sqrt{\epsilon}}{T+1}\right)$, $\theta \equiv \theta(\ker A, \ker B)$

Proof

$$A = H_{in} + H_{out}$$

$$= (\pi_1^{(1)} + \sum_{j \in A} \pi_j^{(1)}) \otimes |0\rangle_d \langle 0| + \pi_1^{(0)} \otimes |T\rangle \langle T|$$

$$\ker A = \text{span} \left\{ |0\rangle, |\psi\rangle \overbrace{|0\rangle \dots |0\rangle}^{\text{ancillas}} \otimes |0\rangle_{cl}, \right. \\ \left. |\psi\rangle \otimes |t\rangle_{cl} \quad 0 < t < T, \right. \\ \left. |1\rangle, |\phi\rangle \otimes |T\rangle_{cl} \right\}$$

$$\therefore \pi_A = |0\rangle_d \langle 0| \otimes \mathbb{1} \otimes (|0\rangle \dots |0\rangle) (\langle 0| \dots \langle 0|) \otimes |0\rangle_d \langle 0| \\ + \sum_{t=1}^{T-1} \mathbb{1} \otimes |t\rangle \langle t| + |1\rangle_d \langle 1| \otimes \mathbb{1} \otimes |T\rangle_d \langle T|$$

$$W \pi_A W^\dagger = \overbrace{|0\rangle_d \langle 0| \otimes \mathbb{1} \otimes (|0\rangle \dots |0\rangle) (\langle 0| \dots \langle 0|) \otimes |0\rangle_d \langle 0|}^{\pi_I} \\ + \sum_{t=1}^{T-1} \mathbb{1} \otimes |t\rangle_d \langle t| \\ + \underbrace{U^\dagger (|1\rangle_d \langle 1| \otimes \mathbb{1}_{[n] \setminus 1}) U}_{\pi_F} \otimes |T\rangle_d \langle T|$$

where $U := U_T U_{T-1} \dots U_2 U_1$.

$$\cos^2 \theta = \max_{\substack{|\xi\rangle \in \ker A \\ |\eta\rangle \in \ker B}} |\langle \eta | \xi \rangle|^2 \quad \text{by Def.}$$

$$= \max_{|\xi\rangle, |\eta\rangle} |\langle \eta | \Pi_A | \xi \rangle|^2 \quad \Pi_A = \text{proj. on } \ker A$$

$$= \max_{|\eta\rangle \in \ker B} \langle \eta | \Pi_A | \eta \rangle \quad \text{Cauchy-Schwarz}$$

$$\text{(saturate } |\langle \eta | \Pi_A | \xi \rangle|^2 \leq \| \Pi_A | \eta \rangle \|^2 \cdot \| |\xi\rangle \|^2 = \langle \eta | \Pi_A | \eta \rangle)$$

$$= \max_{|\eta\rangle \in \ker B} \langle \eta | W^\dagger (W \Pi_A W^\dagger) W | \eta \rangle$$

$$= \max_{|\eta'\rangle \in \ker W B W^\dagger} \langle \eta' | \left(\Pi_I \otimes \mathbb{1}_d + \Pi_F \otimes \mathbb{1}_d + \sum_{t=1}^{T-1} \mathbb{1} \otimes |t\rangle\langle t| \right) | \eta' \rangle$$

$$= \frac{1}{T+1} \max_{|\varphi\rangle} \sum_{t=0}^T \langle \varphi | \langle t | \left(\text{---} \right) \sum_{t'=0}^T |\varphi\rangle |t'\rangle$$

$$W B W^\dagger = \mathbb{1} \otimes E \Rightarrow |\eta\rangle = |\varphi\rangle \otimes \frac{1}{\sqrt{T+1}} \sum_{t=0}^T |t\rangle$$

$$= \frac{1}{T+1} \max_{|\varphi\rangle} \left(\langle \varphi | (\Pi_I + \Pi_F) | \varphi \rangle + T - 1 \right)$$

$$= \frac{1}{T+1} \max_{|\varphi\rangle} \left(\langle \varphi | 2\mathbb{1} - \Pi_I^\perp - \Pi_F^\perp | \varphi \rangle + \frac{T-1}{T+1} \right)$$

$$\leq \frac{2 - 2 \sin^2 \frac{\vartheta}{2}}{T+1} + \frac{T-1}{T+1} \quad \text{by Kitaev Lemma}$$

$$= \frac{1 + \cos \vartheta}{T+1} + \frac{T-1}{T+1}$$

where $\vartheta = \theta(\text{supp } \Pi_I, \text{supp } \Pi_F)$.

$$\cos^2 \vartheta = \max_{\substack{|\eta\rangle \in \text{supp } \mathcal{T}_I \\ |\xi\rangle \in \text{supp } \mathcal{T}_F}} |\langle \eta | \xi \rangle|^2$$

$$= \max_{|\omega\rangle, |\varphi\rangle} \left| \underbrace{(\langle 0 | \langle \omega | \langle 0 | \dots \langle 0 |)}_{= |\psi_0\rangle} \cdot (U^\dagger |1\rangle_1 |\varphi\rangle) \right|^2$$

$$\mathcal{T}_I = |0\rangle_1 \langle 0| \otimes \mathbb{1} \otimes (|0\rangle \dots |0\rangle) (\langle 0| \dots \langle 0|)$$

$$\Rightarrow |\eta\rangle = |0\rangle_1 |\omega\rangle |0\rangle \dots |0\rangle$$

$$\mathcal{T}_F = U^\dagger (|1\rangle_1 \langle 1| \otimes \mathbb{1}) U$$

$$\Rightarrow |\xi\rangle = |1\rangle_1 |\varphi\rangle$$

$$= \max_{|\omega\rangle, |\varphi\rangle} (\langle \psi_0 | U^\dagger) |1\rangle_1 \langle 1| \otimes |\varphi\rangle \langle \varphi| (U |\psi_0\rangle)$$

$$\leq \max_{|\omega\rangle} \langle \psi_T | \mathcal{T}_1^{(1)} | \psi_T \rangle$$

$$\leq \varepsilon. \quad \text{NO instance!}$$

$$\therefore \cos^2 \theta \leq \frac{1 + \cos \vartheta}{T+1} + \frac{T-1}{T+1} \leq 1 - \frac{1 - \sqrt{\varepsilon}}{T+1}$$

$$\sin^2 \frac{\theta}{2} = \frac{1 - \sqrt{\cos^2 \theta}}{2} \geq \frac{1 - \sqrt{\varepsilon}}{4(T+1)} \quad \text{Taylor expand } \sqrt{1-x} \geq 1 - \frac{x}{2}$$

Summary:

$$\lambda_{\min}(A|_{\text{supp}A}) \geq 1$$

$$\lambda_{\min}(B|_{\text{supp}B}) \geq \Omega(T^{-2})$$

$$\sin^2 \frac{\theta}{2} \geq \frac{1 - \sqrt{\varepsilon}}{4(T+1)}$$

Kitaev's Lemma \Rightarrow for NO instance have

$$H = A + B \geq \Omega\left(\frac{1 - \sqrt{\varepsilon}}{T^3}\right)$$

i.e. have lower bound on λ_{\min} :

$$\lambda_{\min}(H) = \Omega\left(\frac{1 - \sqrt{\varepsilon}}{T^3}\right) \quad \text{as required.}$$

We have proven:

$$\text{YES} \Rightarrow \lambda_{\min}(H) \leq \frac{\varepsilon}{T+1} =: \alpha$$

$$\text{NO} \Rightarrow \lambda_{\min}(H) \geq \Omega\left(\frac{1 - \sqrt{\varepsilon}}{T^3}\right) =: \beta$$

$$\text{Taking } \varepsilon = \Omega\left(\frac{1}{T^2}\right) \Rightarrow \beta - \alpha = \Omega\left(\frac{1}{T^3}\right)$$

\therefore Have proven reduction from QMA to Local Hamiltonian ...

... except that clock register has dimension $T+1$

→ H not on qudits with const. local dim. d .

Could split clock into $\log T$ qubits

= binary clock ($\mathbb{C}^T \cong (\mathbb{C}^2)^{\otimes \log T}$)

→ local terms $\log T$ -local, not k -local (k const).

(Example of general phenomenon: can always trade off locality against local dimension.)