III. Quantum Many-body Physics

Have seen that ground states of quantum many-body systems can be very complex.

So complex, finding them is difficult even for a quantum computer (QMA-hard)...

... or for any other physical process, if you believe the quantum Church-Turing thesis.

What makes these ground states so complex?

Part (but only part!) of the answer is that they are highly non-locally entangled:

$$\frac{1}{\sqrt{T}} \sum_t |\psi_t\rangle\, |t\rangle$$

n qubits ($|\psi_t\rangle$) T qubits ($|t\rangle$)

computational & clock registers in history state maximally entangled (at least assuming $\langle\psi_t|\psi_{t'}\rangle = \delta_{tt'}$, though this won't always be the case).

What characterizes Hamiltonians with complex g.s.'s?
$\longrightarrow$ major open question in condensed matter physics!

Can we get clues from our QMA-hard Hamiltonians?

By construction, promise gap between YES/NO instances

$$\lambda_{min}(H_{YES}) - \lambda_{min}(H_{NO}) = \Omega(1/poly).$$

In fact, using same tools, can also show spectral gap of our QMA Hamiltonians is

$$\Delta(H) = \lambda_1(H) - \lambda_0(H) = \Omega\left(\frac{1}{poly}\right).$$

$\Rightarrow$ as system size $n \to \infty$, $\Delta(H) \longrightarrow 0$.

$\therefore$ System is critical (at phase transition).

Expect critical systems to have long-range correlations $\approx$ entanglement in g.s.

Could this be reason behind QMA-hardness?

Could it be non-critical systems (i.e. those with uniform gap $\Delta(H) = \Omega(1)$) have simple ground states?

The relationships between these properties — spectral gap, correlations, entanglement, computational complexity, phase transitions entropy area laws... — are some of <u>the</u> most intriguing and difficult questions in contemporary theoretical physics.

It's a topic frought with challenges for anyone interested in mathematical rigour:

Many "folk-lore" beliefs in condensed matter physics are widely espoused and used in the published literature, yet are false in general (counterexamples!).

There is disagreement even about what the correct definitions of many of the fundamental concepts & quantities should be.

(But you should cut the condensed matter physicists some slack — they are trying to understand the extremely complex systems encountered in the real world, not artificial toy problems encountered in mathematics!)

QIT ideas (like Hamiltonian complexity) are playing a part in tackling things rigorously
⟶ so let's dive in!

Notation & Definitions

$A, B \in \mathcal{B}(\mathcal{H})$

- $[A, B] = AB - BA$
- $\|A\| = \|A\|_\infty := \sup_{|\psi\rangle \in \mathcal{H}} \frac{\|A|\psi\rangle\|}{\||\psi\rangle\|}$

  operator norm (Schatten ∞-norm)

  unitarily invariant: $\|UAU^\dagger\| = \|A\|$
- $\|A\|_1 := \mathrm{tr}\,|A| = \mathrm{tr}\sqrt{A^\dagger A}$

  trace norm (Schatten 1-norm)

  unitarily invariant; note for $A \geq 0$, $\|A\|_1 = \mathrm{tr}\,A$
- $\|A\|_p := (\mathrm{tr}\,|A|^p)^{1/p} = (\mathrm{tr}\,(A^\dagger A)^{p/2})^{1/p}$

  Schatten p-norm
- Recall all norms satisfy triangle inequality:

  $\|A+B\| \leq \|A\| + \|B\|$
- $\langle A, B\rangle = \mathrm{tr}\,(A^\dagger B)$

  Hilbert-Schmidt inner product
- $\|A\|_\infty = \sup_{\substack{B \in \mathcal{B}(\mathcal{H}) \\ \|B\|_1 = 1}} |\langle B, A\rangle|$

  $\|A\|_1 = \sup_{\substack{B \in \mathcal{B}(\mathcal{H}) \\ \|B\|_\infty = 1}} |\langle B, A\rangle|$

  Duality
- Every norm defines a distance measure $\|A - B\|$.

Operational significance

- States ρ, σ.
  How well can we distinguish them?
  Only thing we can do in QM is measure!

  Max difference in outcome probabilities for projective measurement P is:

$$\sup_P |\operatorname{tr}(P\rho) - \operatorname{tr}(P\sigma)| = \sup_P \operatorname{tr} P(\rho - \sigma)$$
$$= \tfrac{1}{2} \|\rho - \sigma\|_1$$

Michaelmas QIT course

- Observables $A, B \geq 0$.
  How well can we distinguish them?

  Max difference in expectation values is:

$$\sup_{\substack{\rho \geq 0 \\ \operatorname{tr}\rho = 1}} |\operatorname{tr}(A\rho) - \operatorname{tr}(B\rho)| = \sup_{\substack{\rho \geq 0 \\ \|\rho\|_1 = 1}} \operatorname{tr}[\rho^\dagger (A - B)]$$
$$= \sup_{\|\rho\|_1 = 1} \langle \rho, A - B \rangle =: \| A - B \|_\infty$$

<u>Trace distance</u>: optimal distinguishability of states.

<u>Operator distance</u>: optimal distinguishability of observables.

Heisenberg Picture (reminder)

Schrödinger picture:

- states evolve as $e^{-iHt}|\psi\rangle$
- measurement operators X fixed

$\longrightarrow$ $Pr(\text{outcome } X) = |\langle\psi(t)|X|\psi(t)\rangle|^2$

$$= |\langle\psi|e^{iHt} X e^{-iHt}|\psi\rangle|^2$$

Heisenberg picture:

- states fixed
- measurement operators X evolve as $e^{iHt} X e^{-iHt}$

$\longrightarrow$ $Pr(\text{outcome } X) = |\langle\psi|X(t)|\psi\rangle|^2$

$$= |\langle\psi|e^{iHt} X e^{-iHt}|\psi\rangle|^2$$

Only physically observable content of QM is probabilities of measurement outcomes

$\longrightarrow$ Heisenberg & Schröding pictures describe <u>same</u> physical theory.

Correlation functions

Many-body system $(\mathbb{C}^d)^{\otimes n}$ (n qudits)
State $|\psi\rangle$.

Observables A, B
(typically on disjoint subsets of qudits)

terminology comes from Feynman diagrams: can be computed by summing all connected diags.

Def (Connected correlation function)

$$C(X,Y) = \langle\psi|AB|\psi\rangle - \langle\psi|A|\psi\rangle\langle\psi|B|\psi\rangle$$

Physical relevance: these are what we can measure in real experiments.

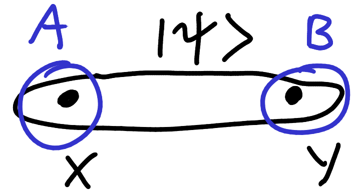
Connected correlation functions measure "quantum correlations".

Can be made somewhat precise:

If state is close to separable (i.e. not entangled), correlation function is close to 0.

Converse is not true.
(e.g. hiding states — beyond scope of course.)

Proposition



$|\psi\rangle$ close to product:

$\| \psi_{XY} - \varphi_X \otimes \varphi_Y \| \leq \varepsilon$ for some φ_X, φ_Y

Normalised observables $\|A_X\|, \|B_Y\| \leq 1$.

$\Rightarrow C(A_X, B_Y) \leqslant O(\varepsilon)$.

Proof

$C(A_X, Y_B)$

$$= \mathrm{tr}(A_X B_Y \psi_{XY}) - \mathrm{tr}(A_X \psi_{XY})\, \mathrm{tr}(B_Y \psi_{XY})$$

$$= \mathrm{tr}(A \otimes B \cdot \varphi_X \otimes \varphi_Y) + \underbrace{\mathrm{tr}[A \otimes B \cdot (\psi_{XY} - \varphi_X \otimes \varphi_Y)]}_{\leq \varepsilon}$$

$$- \left(\mathrm{tr}[A \varphi_X] + \underbrace{\mathrm{tr}[A (\psi_X - \varphi_X)]}_{\leq \varepsilon} \right)$$

$$\times \left(\mathrm{tr}[B \varphi_Y] + \underbrace{\mathrm{tr}[B (\psi_Y - \varphi_Y)]}_{\leq \varepsilon} \right)$$

$$\leqslant \varepsilon \left(\mathrm{tr}(A \varphi_X) + \mathrm{tr}(B \varphi_Y) + \varepsilon \right)$$

(using contractivity of trace-distance under partial trace)

$\square$