Lieb - Robinson bounds

In relativistic QM, causality implies that space-like separated observables commute, and continue to commute as they evolve, until their light-cones intersect.

In non-relativistic QM, this is <u>not</u> true.

Let
$$A_X \equiv A_X \otimes \mathbb{1}_{rest}$$
$$B_Y \equiv B_Y \otimes \mathbb{1}_{rest}$$
act non-trivially on disjoint subsets X, Y respectively of a many-body system, so

$$[A_X, B_Y] = 0.$$

In general, $\forall\, t > 0$,

$$[A_X(t), B_Y(t)] \neq 0.$$

Does causality break down completely in non-relativistic QM?

No! Lieb-Robinson bounds show an approximate version of causality does still hold.

There exists a finite speed of propagation of information in a many-body system, & observables <u>approximately</u> commute until their "light cones" intersect.

(Surprisingly, this dynamical result is very useful in proving static properties ...)

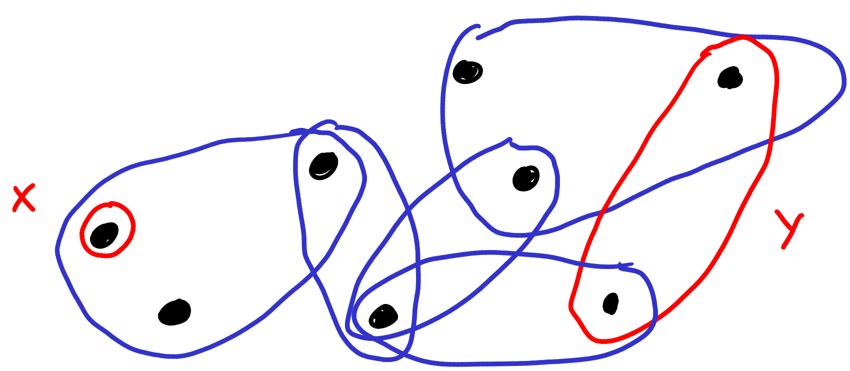
<u>Def</u> (interaction distance)

k-local Hamiltonian $H = \sum_Z h_Z$

subsets of qudits X, Y

$$d(X,Y) := \min \left| \left\{ Z_i : \begin{array}{l} X \cap Z_1 \neq \emptyset \\ Z_i \cap Z_{i+1} \neq \emptyset \\ Z_n \cap Y \neq \emptyset \end{array} \right\}_{i=1\ldots n} \right|$$

I.e. interaction distance is min # "hops" along interactions to get from X to Y.



$d(X,Y) = 3$

Theorem (Lieb-Robinson)

k-local Hamiltonian $H = \sum_{Z} h_Z$ such that $\exists\ \mu, s > 0$ s.t. $\forall$ qudits i : $\sum_{Z \ni i} \| h_Z \| \leq s e^{-\mu}$.

A_X, B_Y operators on subsets of qudits X, Y.

Then

$$\| [A_X(t), B_Y] \| \leq 2 \| A_X \| \cdot \| B_Y \| \min(|X|, |Y|)\, e^{-\mu d(X,Y)} (e^{2kst} - 1).$$

Proof

Let $H = H_Y + H_{Y^c}$

where $H_Y = \sum_{Z \cap Y \neq \emptyset} h_Z$, $\quad H_{Y^c} = \sum_{Z \cap Y = \emptyset} h_Z$.

Note $[H_{Y^c}, B_Y] = 0$.

Write $f(t) = [A(t), B] \in \mathcal{B}(\mathcal{H})$.

$$\begin{aligned}
\frac{d}{dt} f(t) &= \frac{d}{dt} [A(t), B] \\
&= \frac{d}{dt} \left(e^{iHt} A e^{-iHt} B - B e^{iHt} A e^{-iHt} \right) \\
&= [\, i [H, A(t)], B] \\
&= i [H_{Y^c}, [A(t), B]] + [\, i [H_Y, A(t)], B] \\
&= i [H_{Y^c}, f(t)] + [\, i [H_Y, A(t)], B]
\end{aligned}$$

This is an inhomogeneous linear ODE for $f(t)$.

$\longrightarrow$ Solve using variation of parameters (<u>Exercise</u>):

$$[A(t), B] = e^{iH_{yc}t} [A(0), B] \, e^{-iH_{yc}t} + \int_0^t e^{iH_{yc}(t-s)} [i[H_y, A(s)], B] \, e^{-iH_{yc}(t-s)} \, ds$$

Taking norms:

$$\| [A(t), B] \|$$

$$\leq \| [A(0), B] \| + \int_0^t \| [i[H_y, A(s)], B] \| \, ds$$

unitary invariance + triangle ineq.

$$\leq \| [A(0), B] \| + 2 \|B\| \int_0^t \| [A(s), H_y] \| \, ds$$

$\| [A, B] \| \leq 2 \|A\| \cdot \|B\|$

$$= \| [A(0), B] \| + 2 \|B\| \sum_{Z \cap Y \neq \emptyset} \int_0^t \| [A(s), h_Z] \| \, ds$$

Let $\quad C_A(Z,t) := \sup_{O_Z} \dfrac{\| [A_X(t), O_Z] \|}{\| O_Z \|} \geq 0$

Note:

- $\| [A(t), B] \| \leq \|B\| \, C_A(Y,t)$.
- $C_A(Z,0) \begin{cases} = 0 & X \cap Z = \emptyset \\ \leq 2\|A\| & X \cap Z \neq \emptyset \end{cases}$

$$= 2\|A\| \, \delta(X,Z)$$

$$\delta(X,Z) = \begin{cases} 0 & X \cap Z = \emptyset \\ 1 & X \cap Z \neq \emptyset \end{cases}$$

Rewriting, we have (from above):

$$\boxed{C_A(Y,t) \leq C_A(Y,0) + 2 \sum_{Z:\, Y \cap Z \neq \emptyset} \|h_Z\| \int_0^t C_A(Z,s)\, ds \quad (*)}$$

$\left(f(x) \leq g(x) \;\Rightarrow\; \sup_x f(x) =: f(x^*) \leq g(x^*) \leq \sup_x g(x) \right)$

Iterating $(*)$: (related to Picard iteration)

$$C_A(Y,t)$$

$$\leq 0 + 2\sum_{Z_1\cap Y\neq\emptyset} \|h_{Z_1}\| \int_0^t ds_1\, C_A(Z_1, s_1)$$

sum over Z_1 (drop "Z_1:" for brevity)

$$\leq 2\sum_{Z_1\cap Y\neq\emptyset} \|h_{Z_1}\| \int_0^t ds_1 \Big(C_A(Z_1, 0) + 2\sum_{Z_2\cap Z_1\neq\emptyset} \|h_{Z_2}\| \int_0^{s_1} ds_2\, C_A(Z_2, s_2) \Big)$$

using $(*)$

$$\leq 2\sum_{Z_1\cap Y\neq\emptyset} \|h_{Z_1}\| \int_0^t 2\|A\|\,\delta(X, Z_1) + 2^2 \sum_{Z_1\cap Y\neq\emptyset} \|h_{Z_1}\| \sum_{Z_2\cap Z_1\neq\emptyset} \|h_{Z_2}\| \int_0^t ds_1 \int_0^{s_1} ds_2\, C_A(Z_2, s_2)$$

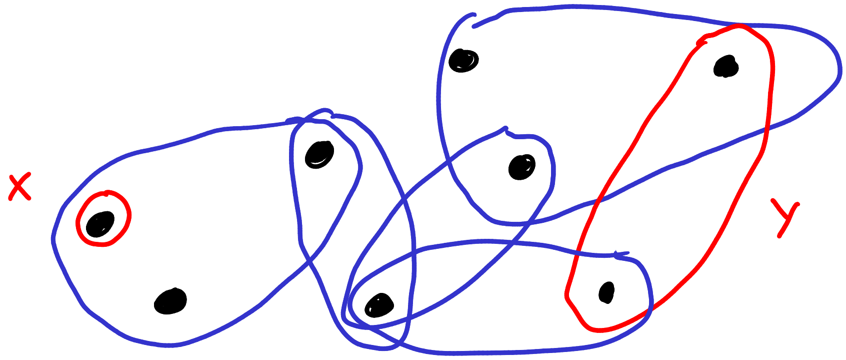
using $C_A(Z, 0) \leq 2\|A\|\,\delta(X, Z)$

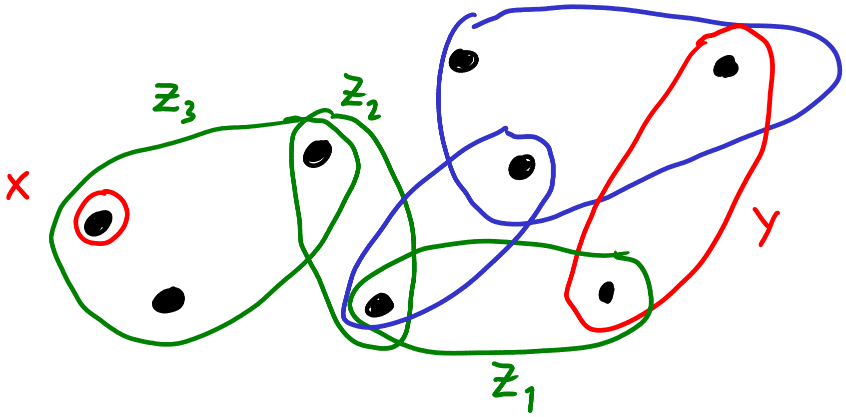
$$\leq 2\|A\|\,(2t) \sum_{\substack{Z_1\cap Y\neq\emptyset \\ Z_1\cap X\neq\emptyset}} \|h_{Z_1}\|$$
$$+ 2^2 \sum_{Z_1\cap Y\neq\emptyset} \|h_{Z_1}\| \sum_{Z_2\cap Z_1\neq\emptyset} \|h_{Z_2}\| \int_0^t ds_1 \int_0^{s_1} ds_2\, 2\|A\|\,\delta(X, Z_2)$$
$$+ 2^3 \sum_{Z_1\cap Y\neq\emptyset} \|h_{Z_1}\| \sum_{Z_2\cap Z_1\neq\emptyset} \|h_{Z_2}\| \sum_{Z_3\cap Z_2\neq\emptyset} \|h_{Z_3}\| \int_0^t ds_1 \int_0^{s_1} ds_2 \int_0^{s_2} ds_3\, C_A(Z_3, s_3)$$

using $(*)$

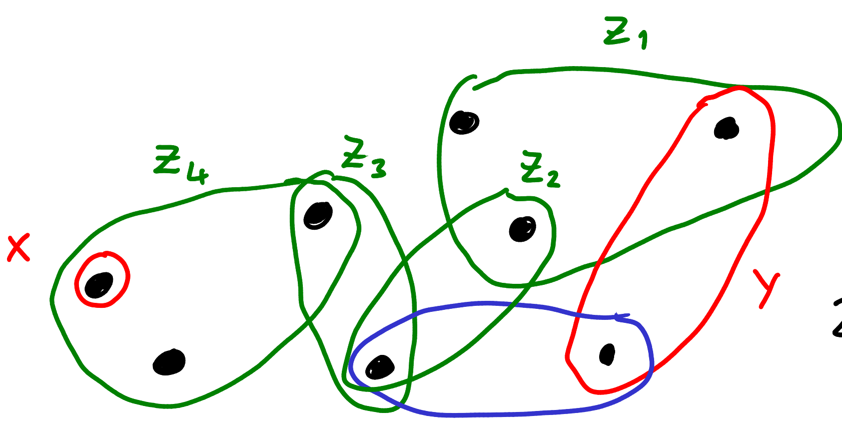
$$
\begin{aligned}
&= 2\,\|A\|\,(2t) \sum_{\substack{Z_1 \cap Y \neq \phi \\ Z_1 \cap X \neq \phi}} \|h_{Z_1}\| \\
&\quad + 2\,\|A\|\,\frac{(2t)^2}{2!} \sum_{Z_1 \cap Y \neq \phi} \|h_{Z_1}\| \sum_{\substack{Z_2 \cap Z_1 \neq \phi \\ Z_2 \cap X \neq \phi}} \|h_{Z_2}\| \\
&\quad + 2\,\|A\|\,\frac{(2t)^3}{3!} \sum_{Z_1 \cap Y \neq \phi} \|h_{Z_1}\| \sum_{Z_2 \cap Z_1 \neq \phi} \|h_{Z_2}\| \sum_{\substack{Z_3 \cap Z_2 \neq \phi \\ Z_3 \cap X \neq \phi}} \|h_{Z_3}\| \\
&\quad + \cdots
\end{aligned}
$$

$$
= 2\,\|A\| \sum_{n=1}^{\infty} \frac{(2t)^n}{n!} \sum_{Z_1 \cap Y \neq 0} \|h_{Z_1}\| \sum_{Z_2 \cap Z_1 \neq \phi} \|h_{Z_2}\| \cdots \sum_{\substack{Z_n \cap Z_{n-1} \neq \phi \\ Z_n \cap X \neq \phi}} \|h_{Z_n}\| . \qquad (**)
$$





$$2\|A\| \frac{(2t)^3}{3!} \|h_{z_1}\| \|h_{z_2}\| \|h_{z_3}\|$$



$$2\|A\| \frac{(2t)^4}{4!} \|h_{z_1}\| \|h_{z_2}\| \|h_{z_3}\| \|h_{z_4}\|$$

Bound each term using assumption on interaction strengths in Thm:

$$\sum_{Z_1 \cap Y \neq 0} \| h_{Z_1} \| \cdots \sum_{Z_{n-1} \cap Z_{n-2} \neq \emptyset} \| h_{Z_{n-1}} \| \sum_{\substack{Z_n \cap Z_{n-1} \neq \emptyset \\ Z_n \cap X \neq \emptyset}} \| h_{Z_n} \|$$

$$\leq \sum_{i_1 \in Y} \sum_{Z_1 \ni i_1} \| h_{Z_1} \| \cdots \sum_{j \in Z_{n-2}} \sum_{Z_{n-1} \ni j} \| h_{Z_{n-1}} \| \sum_{k \in Z_{n-1}} \sum_{\substack{Z_n \ni k \\ Z_n \cap X \neq \emptyset}} \| h_{Z_n} \|$$

$$\leq \sum_{i \in Y} \sum_{Z_1 \ni i} \| h_{Z_1} \| \cdots \sum_{j \in Z_{n-2}} \sum_{Z_{n-1} \ni j} \| h_{Z_{n-1}} \| \sum_{k \in Z_{n-1}} s e^{-\mu}$$

$$\leq \sum_{i \in Y} \sum_{Z_1 \ni i} \| h_{Z_1} \| \cdots \sum_{j \in Z_{n-2}} \sum_{Z_{n-1} \ni j} \| h_{Z_{n-1}} \| \, k s e^{-\mu}$$

$$\leq \sum_{i \in Y} \sum_{Z_1 \ni i} \| h_{Z_1} \| \cdots \sum_{j \in Z_{n-2}} k s^2 e^{-2\mu}$$

$$\leq \sum_{i \in Y} \sum_{Z_1 \ni i} \| h_{Z_1} \| \cdots k^2 s^2 e^{-2\mu}$$

$$\leq \cdots$$

$$\leq \begin{cases} \sum_{i \in Y} (k s e^{-\mu})^n & \exists \text{ path } Y \to X \text{ of length } n \\ 0 & \nexists \text{ path} \end{cases}$$

$$\leq |Y| \, (ks)^n \, e^{-\mu d(X,Y)}$$

Inserting bound in $(**)$, we have:

$$C_A(Y,t) \le 2\|A\| \sum_{n=1}^{\infty} \frac{(2t)^n}{n!} |Y| (ks)^n e^{-\mu d(X,Y)}$$
$$= 2\|A\| \, |Y| \, e^{-\mu d(X,Y)} (e^{2kst} - 1).$$

Note we could equally well have summed paths in $(**)$ from other end, to get:

$$C_A(Y,t) \le 2\|A\| \min(|X|,|Y|) e^{-\mu d(X,Y)} (e^{2kst} - 1).$$

Thm follows from $\|[A(t), B]\| \le \|B\| C_A(Y,t)$. $\square$

<u>Exercise</u>

Generalise L-R bounds to quasi-local Hamiltonians (exponentially-decaying interaction strength).

Corollary

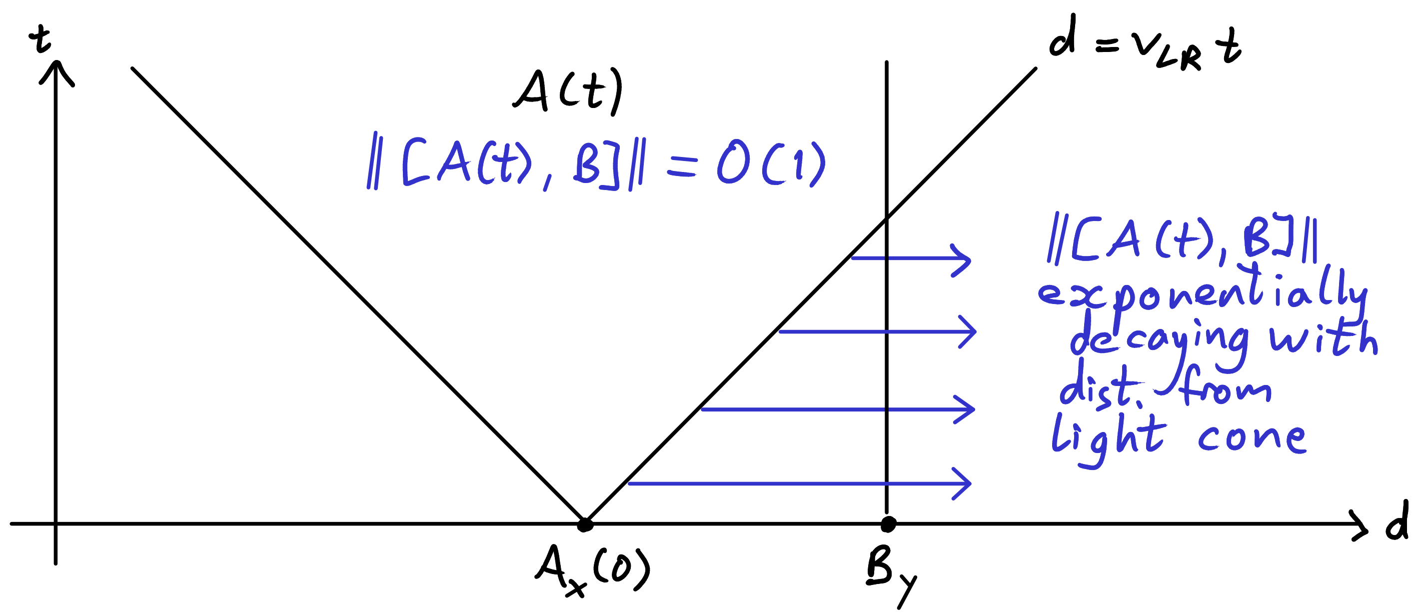
For k-local Hamiltonian as in Theorem,

$\exists\, v_{LR}$ depending only on μ, s s.t.

$$\| [A_X(t), B_Y] \| \leq \min(|X|, |Y|)\ \|A\|\ \|B\|\ e^{-\mu(d(X,Y) - v_{LR} t)}$$

Proof

Follows immediately from Thm, taking $v_{LR} = \dfrac{2ks}{\mu}$.



$\forall t > 0$, $A_X(t)$ acts non-trivially on entire system.

But L-R bound $\Rightarrow$ can <u>approximate</u> $A(t)$ by observable that only acts within light cone.

For proof, will need

<u>Lemma</u> (Twirling)

$X_{AB} \in B(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})$

$$tr_B[X_{AB}] \otimes \mathbb{1}_B = d_B \int dU \; \mathbb{1} \otimes U_B \cdot X_{AB} \cdot \mathbb{1} \otimes U_B^\dagger$$

Haar measure on $SU(d_B)$

<u>Proof</u>

Note: for any operator Y:

$$Y = \sum_{nm} \langle m | Y | n \rangle \cdot |m\rangle\langle n| .$$

Decompose X_{AB} in product basis:

$$X_{AB} = \sum_{ijkl} x_{ijkl} \; \underset{A}{|i\rangle\langle j|} \otimes \underset{B}{|k\rangle\langle l|}$$

so
$$tr_B X_{AB} = \sum_{ij} \Big(\sum_k x_{ijkk} \Big) |i\rangle\langle j|$$

$$= \sum_{\substack{ijkl \\ mn}} x_{ijkl} \, \delta_{im} \, \delta_{jn} \, \delta_{kl} \; |m\rangle\langle n| \qquad (*)$$

Let $V_B \in SU(d_B)$.

$$\mathbb{1}\otimes V_B \left(d_B \int dU \; \mathbb{1}\otimes U_B \cdot (|i\rangle\langle j| \otimes |k\rangle\langle \ell|) \cdot \mathbb{1}\otimes U_B^\dagger \right) \mathbb{1}\otimes V_B^\dagger$$

$$= d_B \int dU \; \mathbb{1}\otimes U_B \cdot (|i\rangle\langle j| \otimes |k\rangle\langle \ell|) \cdot \mathbb{1}\otimes U_B^\dagger$$

by invariance of Haar measure

Thus

$$\left[\mathbb{1}\otimes V_B \,,\, d_B \int dU \; \mathbb{1}\otimes U_B \cdot (|i\rangle\langle j| \otimes |k\rangle\langle \ell|) \cdot \mathbb{1}\otimes U_B^\dagger \right] = 0$$

$$\Rightarrow \; d_B \int dU \; \mathbb{1}\otimes U_B \cdot (|i\rangle\langle j| \otimes |k\rangle\langle \ell|) \cdot \mathbb{1}\otimes U_B^\dagger = Y \otimes \mathbb{1}$$

by Schur's Lemma + fact that unitaries span full algebra $GL(d_B)$.

$$\langle m | Y | n \rangle = \frac{1}{d_B} \operatorname{tr}\left(|n\rangle\langle m| \otimes \mathbb{1} \cdot Y \otimes \mathbb{1} \right)$$

$$= \frac{1}{d_B} \operatorname{tr}\left(|n\rangle\langle m| \otimes \mathbb{1} \cdot d_B \int dU \; \mathbb{1}\otimes U_B \cdot (|i\rangle\langle j| \otimes |k\rangle\langle \ell|) \cdot \mathbb{1}\otimes U_B^\dagger \right)$$

$$= \delta_{im} \, \delta_{jn} \int dU \; \operatorname{tr}\left(U |k\rangle\langle \ell| U^\dagger \right)$$

$$= \delta_{im} \, \delta_{jn} \, \delta_{k\ell}$$

by unitary invariance of trace
+ normalisation of Haar measure $\int dU = 1$

Lemma follows from (*) by linearity. $\square$

Corollary

$\exists\ A_{X(\ell)}(t)$ acting non-trivially only on $X(\ell) = \{ i : d(i, X) \leq v_{LR} t + \ell \}$

$$\| A_X(t) - A_{X(\ell)}(t) \| \leq |X| \, \|A_X\| \, e^{-\mu \ell}$$

Proof

Let $A_{X(\ell)}(t) := \int dU \, U A_X(t) U^{\dagger}$

where U acts on $X(\ell)^c$.

$$A_{X(\ell)}(t) = \mathrm{tr}_{X(\ell)^c} [A_X(t)] \otimes \frac{\mathbb{1}}{d_B}$$ by Lemma

$\| A_X(t) - A_{X(\ell)}(t) \|$

$= \int dU \, \| [U, A_X(t)] \|$ unitary invariance of operator norm

$\leq \int dU \, |X| \, \|A\| \, e^{-\mu \ell}$ Lieb-Robinson $+ \|U\| = 1$

$\leq |X| \, \|A\| \, e^{-\mu \ell}$ $\square$