# Exponential decay of correlations

Condensed matter physics folk-lore:
spectral gap $\Leftrightarrow$ exponential decay of correlations (with distance).

(In relativistic QM, this is Fredenhagen's Thm.)

Only problem is, this folk-lore is false! (in non-relativistic many-body QM).

There are examples of gapless many-body systems with exp. decay of correlations:
exp. decay $\not\Rightarrow$ spectral gap.

What about converse?

Widely believed, but not proven until 2004/2005 by Matt Hastings using Lieb-Robinson techniques.

<u>Thm</u> (Exponential clustering)

k-Local Hamiltonian $H$ with:

- spectral gap $\Delta > 0$
- unique ground state $|\phi_0\rangle$
- Lieb-Robinson constants $\mu, s$

$A_X$, $B_Y$ operators on subsets of qudits $X, Y$.

Then

$$\langle \phi_0 | A_X B_Y | \phi_0 \rangle - \langle \phi_0 | A_X | \phi_0 \rangle \langle \phi_0 | B_Y | \phi_0 \rangle$$
$$\leq O\left( \|A\| \, \|B\| \min(|X|, |Y|) \, e^{-\tilde{\mu}\, d(X,Y)} \right)$$

where $\tilde{\mu} = \dfrac{\mu}{1 + 4ks/\Delta}$.

We will prove this result rigorously shortly, but first we will spend a little time discussing the intuition behind the proof.

In proving that time-evolved observables can be approximated by observables with compact support, it was intuitively clear that Lieb-Robinson bounds should play a role (though it required an elegant mathematical trick to make the connection), as the result itself concerned time-evolution.

Exponential decay of correlations concerns static properties of the system in its ground state.
$\rightarrow$ Where do Lieb-Robinson bounds come in?

Consider matrix elements of an operator $A$ in energy eigenbasis (i.e. eigenbasis of Hamiltonian):

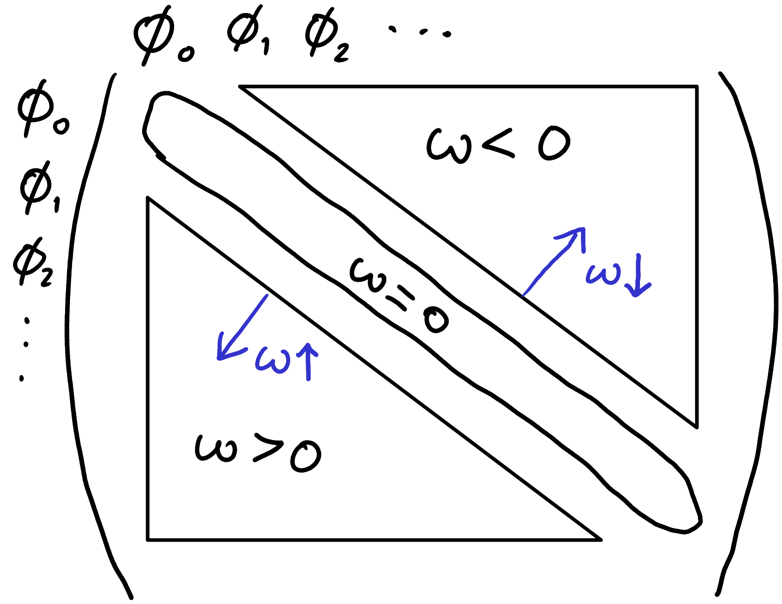
$$A_{ij} = \langle \phi_i | A | \phi_j \rangle \quad , \qquad H|\phi_i\rangle = E_i |\phi_i\rangle$$

Matrix elements evolve as:

$$\begin{aligned} A_{ij}(t) &= \langle \phi_i | A(t) | \phi_j \rangle \\ &= \langle \phi_i | e^{iHt} A e^{-iHt} | \phi_j \rangle \\ &= A_{ij} \, e^{i \overbrace{(E_i - E_j)}^{} t} \end{aligned}$$

$\hookrightarrow$ frequency $\omega_{ij}$

$A(t) =$




Can "select" different "sectors" of matrix elements of $A$ in energy eigenbasis using Fourier analysis of time-evolved operator $A(t)$:

1. Fourier transform $A(t)$
2. Apply frequency filter
3. Inverse Fourier transform
4. Evaluate result at $t = 0$

Let's see how this might work for ground state correlations...

Assume $\langle \phi_0 | A | \phi_0 \rangle = 0$.

(In fact, will see later that this is wlog.)

Then

$$\langle \phi_0 | AB | \phi_0 \rangle - \langle \phi_0 | A | \phi_0 \rangle \langle \phi_0 | B | \phi_0 \rangle$$

$$= \langle \phi_0 | AB | \phi_0 \rangle$$

$$= (1, 0, \cdots, 0) \begin{pmatrix} 0 & \text{---} \\ & A_{ij} \end{pmatrix} \begin{pmatrix} | & B_{ij} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$= (1, 0, \cdots, 0) \underbrace{\begin{pmatrix} 0 & \omega < 0 \\ 0 & \end{pmatrix}}_{A_{\omega<0}} \begin{pmatrix} B_{ij} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

So only negative frequency sector of $A$ is relevant for correlation function.

To select -ve frequency sector of $A$:

1. Fourier transform $A(t)$: $\hat{A}(\omega)$

2. Apply frequency filter: $\hat{A}(\omega)\,\Theta(\omega)$

   Step function $\Theta(\omega) = \begin{cases} 1 & \omega < 0 \\ 0 & \omega > 0 \end{cases}$

3. Inverse Fourier transform: $A(t) * \hat{\Theta}(t)$

$$= \int dt'\, A(t')\,\hat{\Theta}(t-t')$$

4. Evaluate result at $t=0$: $\int dt'\, A(t')\,\hat{\Theta}(-t')$

$$\longrightarrow A_{\omega<0} = \int dt\, A(t)\,\hat{\Theta}(-t)$$

How can we relate this to Lieb-Robinson?

Note

$$\langle \phi_0 | B\, A_{\omega<0} | \phi_0 \rangle$$

$$= (1, 0, \cdots, 0) \begin{pmatrix} & & \\ & B_{ij} & \\ & & \end{pmatrix} \begin{pmatrix} 0 & & \omega<0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$= 0$$

$$\Rightarrow \langle \phi_0 | A B | \phi_0 \rangle = \langle \phi_0 | [A_{\omega<0}, B] | \phi_0 \rangle .$$

$$= \int dt \, \langle \phi_0 | [A(t), B] | \phi_0 \rangle \, \hat{\Theta}(-t)$$

For small values of $t$, $[A(t), B]$ is small by L-R.

Problem: integral is dominated by large $t$ component where L-R bound is trivial.

$\longrightarrow$ Introduce Gaussian envelope to suppress large $t$ tail:

$$\int dt \, \langle \phi_0 | [A(t), B] | \phi_0 \rangle \, \hat{\Theta}(-t) \, e^{-\alpha t^2}$$

(compare with $A_{\omega<0}$, above)

Gaussian envelope of width $\sim 1/\alpha$

But now we are no longer exactly selecting -ve frequency component:

Fourier-transforming:

$$\mathcal{F}\left( \langle \phi_0 | [A(t), B] | \phi_0 \rangle \, \hat{\Theta}(-t) \, e^{-\alpha t^2} \right)$$

$$= \langle \phi_0 | [\tilde{A}(\omega), B] | \phi_0 \rangle \, \mathcal{F}\left( \hat{\Theta}(-t) \, e^{-\alpha t^2} \right)$$

$$\simeq \langle \phi_0 | [\tilde{A}(\omega), B] | \phi_0 \rangle \left( \Theta(-\omega) * e^{-\omega^2/4\alpha} \right)$$

convolution of step function with Gaussian of width $\sim \alpha$

To control integral of commutator, want fast-decaying Gaussian ($\alpha$ big).

For good approximation to -ve frequency filter, want fast-decaying Fourier transform, i.e. want flat Gaussian ($\alpha$ small).

$\longrightarrow$ Trade off control of long-time tail of integral against accuracy of frequency filter to get optimal bound on correlation function.

Now we make this intuition rigorous...

Note Fourier Transform convention in operation:

$$\hat{f}(\omega) = \mathcal{F}[f(t)] := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\, dt$$

$$f(t) = \mathcal{F}^{-1}[\hat{f}(\omega)] := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} \hat{f}(\omega)\, d\omega$$

so

$$\mathcal{F}[\mathcal{F}[f(t)]] = f(-t).$$

**Thm** (Exponential clustering - Hastings)

$k$-local Hamiltonian $H$ with:

- spectral gap $\Delta > 0$
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- Lieb-Robinson constants $\mu, s$

$A_X$, $B_Y$ operators on subsets of qudits $X, Y$.

Then

$$\langle \phi_0 | A_X B_Y | \phi_0 \rangle - \langle \phi_0 | A_X | \phi_0 \rangle \langle \phi_0 | B_Y | \phi_0 \rangle$$

$$\leq O\left( \|A\| \, \|B\| \min(|X|, |Y|) \, e^{-\tilde{\mu} d(X,Y)} \right)$$

where $\tilde{\mu} = \dfrac{\mu}{1 + 4ks/\Delta}$ .

For proof, will need following:

**Lemma**

For any $\alpha > 0$, $E \in \mathbb{R}$,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\Theta}(-t) e^{-iEt} e^{-\alpha t^2} dt = \begin{cases} 1 + O(e^{-\Delta^2/4\alpha}) & E \geq \Delta \\ O(e^{-\Delta^2/4\alpha}) & E \leq -\Delta \end{cases}$$

where $\hat{\Theta}(t) = \sqrt{\frac{\pi}{2}} \left( \delta(t) + \frac{i}{\pi t} \right)$ is the Fourier transform of the step function

$$\Theta(\omega) = \begin{cases} 1 & \omega \leq 0 \\ 0 & \omega > 0 . \end{cases}$$

We defer the proof of this Lemma to later.

<u>Proof</u> (of Thm)

Wlog can take $\langle\phi|A|\phi\rangle = \langle\phi|B|\phi\rangle = 0$.

(Let $A' = A - \alpha \mathbb{1}$, $B' = B - \beta \mathbb{1}$
where $\alpha = \langle\phi|A|\phi\rangle$, $\beta = \langle\phi|B|\phi\rangle$.
$\langle\phi|A'B'|\phi\rangle = \langle\phi|AB|\phi\rangle - \langle\phi|A|\phi\rangle\langle\phi|B|\phi\rangle$.)

Define

$$\tilde{A}_x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\Theta}(-t)\, e^{-\alpha t^2} A_x(t)\, dt .$$

$$\langle\phi|A_x B_y|\phi\rangle - \overbrace{\cancel{\langle\phi|A_x|\phi\rangle}}^{=0}\langle\phi|B_y|\phi\rangle$$

$$\begin{aligned}
&= \langle\phi|[\tilde{A}_x, B_y]|\phi\rangle \\
&\quad + \langle\phi|A_x B_y|\phi\rangle - \langle\phi|[\tilde{A}_x, B_y]|\phi\rangle
\end{aligned}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\Theta}(-t)\, e^{-\alpha t^2} \langle\phi|[A_x(t), B_y]|\phi\rangle \tag{1}$$

$$- \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\Theta}(-t)\, e^{-\alpha t^2} \langle\phi|[A_x(t), B_y]|\phi\rangle - \langle\phi|A_x B_y|\phi\rangle \right) \tag{2}$$

## Term (1)

Lieb–Robinson bound controls integrand for small $t$, Gaussian tail controls large $t$:

$$\left| \int_{-\infty}^{\infty} \hat{\Theta}(-t)\, e^{-\alpha t^2} \langle \phi_0 | [A_X(t), B_Y] | \phi_0 \rangle \, dt \right|$$

$$= \left| \int_{-\infty}^{\infty} \sqrt{\frac{\pi}{2}} \left( \delta(-t) - \frac{i}{\pi t} \right) e^{-\alpha t^2} \langle \phi_0 | [A_X(t), B_Y] | \phi_0 \rangle \, dt \right|$$

$$\leq \sqrt{\frac{\pi}{2}} \int_{-\infty}^{\infty} \left( \delta(t) + \frac{1}{\pi t} \right) e^{-\alpha t^2} \, \| [A_X(t), B_Y] \| \, dt$$

$$\leq \sqrt{\frac{\pi}{2}} \int_{|t| \leq cd} \left( \delta(t) + \frac{1}{\pi t} \right) \| [A_X(t), B_Y] \| \, dt + \sqrt{2\pi} \, \|A\| \, \|B\| \int_{|t| \geq cd} e^{-\alpha t^2} dt$$

where $d = d(X, Y)$

$$\leq \sqrt{\frac{\pi}{2}} \, \|A\| \cdot \|B\| \min(|X|, |Y|) \, e^{-\mu d} \int_{|t| \leq cd} \left( \delta(t) + \frac{1}{\pi t} \right) \left( e^{2ks t} - 1 \right) dt$$

$$+ \sqrt{2\pi} \, \|A\| \, \|B\| \, \frac{2 e^{-\alpha c^2 d^2}}{\sqrt{\alpha}\, c d}$$

using Lieb–Robinson for 1st term, & simple Gaussian tail bound for 2nd term:

$$\int_a^{\infty} e^{-x^2/2} dx < \int_a^{\infty} \frac{x}{a} e^{-x^2/2} dx = \frac{e^{-a^2}}{a} .$$

$$= O\left(\|A\|\,\|B\|\left(\min(|X|,|Y|)\, e^{-\mu d} e^{2kscd} + e^{-\alpha c^2 d^2}\right)\right)$$

using $\displaystyle\int_{-a}^{a} \frac{e^x - 1}{x}\, dx < \int_{-a}^{a} \frac{x e^x}{x}\, dx < e^a$

$$\leq O\left(\|A\|\,\|B\| \min(|X|,|Y|) e^{-d(\mu + c[\Delta/2 - 2ks)]}\right)$$

choosing $\alpha = \dfrac{\Delta}{2cd}$

$$= O\left(\|A\|\,\|B\| \min(|X|,|Y|) e^{-\tilde{\mu} d}\right)$$

where $\tilde{\mu} = \dfrac{\mu}{1 + 4ks/\Delta}$

choosing $c = \dfrac{\mu}{2ks + \Delta/2}$ .

Term (2)

Bound using Lemma:

Let $P_0 := |\phi_0\rangle\langle\phi_0|$,
$|\phi_n\rangle$ be energy $E_n$ eigstate of $H$
(i.e. $H|\phi_n\rangle = E_n|\phi_n\rangle$).

Note $\langle\phi_0|A P_0 B|\phi_0\rangle = 0$ by assumption.

We have

$$\langle\phi|[A(t), B]|\phi\rangle$$

$$= \langle\phi_0|A(t)(\mathbb{1} - P_0)B|\phi_0\rangle - \langle\phi_0|B(\mathbb{1} - P_0)A(t)|\phi_0\rangle$$

$$= \sum_{n\neq 0}\Big(\langle\phi_0|e^{iHt}Ae^{-iHt}|\phi_n\rangle\langle\phi_n|B|\phi_0\rangle - \langle\phi_0|B|\phi_n\rangle\langle\phi_n|e^{iHt}Ae^{-iHt}|\phi_0\rangle\Big)$$

$$= \sum_{n\neq 0}\Big(\langle\phi_0|A|\phi_n\rangle\langle\phi_n|B|\phi_0\rangle e^{-it(E_n-E_0)} - \langle\phi_0|B|\phi_n\rangle\langle\phi_n|A|\phi_0\rangle e^{it(E_n-E_0)}\Big) \quad (*)$$

Thus

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{\Theta}(-t)\, e^{-\alpha t^2} \langle \phi | [A_X(t),\, B_Y] | \phi \rangle$$

$$= \sum_{n\neq 0} \Bigg( \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{\Theta}(-t)\, e^{-\alpha t^2} e^{-it\underbrace{(E_n - E_0)}_{\geq \Delta}} \langle \phi_0 | A | \phi_n \rangle \langle \phi_n | B | \phi_0 \rangle$$

$$- \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{\Theta}(-t)\, e^{-\alpha t^2} e^{-it\underbrace{(E_0 - E_n)}_{\leq -\Delta}} \langle \phi_0 | B | \phi_n \rangle \langle \phi_n | A | \phi_0 \rangle \Bigg)$$

$$= \sum_{n\neq 0} \Bigg( \langle \phi_0 | A | \phi_n \rangle \langle \phi_n | B | \phi_0 \rangle \left[ 1 + O(e^{-\Delta^2/4\alpha}) \right]$$

$$- \langle \phi_0 | B | \phi_n \rangle \langle \phi_n | A | \phi_0 \rangle \left[ O(e^{-\Delta^2/4\alpha}) \right] \Bigg)$$

using Lemma & <u>assumption on spectral gap</u>

$$= \langle \phi_0 | A (\mathbb{1} - P_0) B | \phi_0 \rangle + O\left( \|A\| \cdot \|B\| \cdot e^{-\Delta^2/4\alpha} \right)$$

$$= \langle \phi_0 | A B | \phi_0 \rangle + O\left( \|A\|\, \|B\|\, e^{-\Delta^2/4\alpha} \right)$$

recall $\langle \phi_0 | A | \phi_0 \rangle = \langle \phi_0 | B | \phi_0 \rangle = 0$
by assumption

Putting bounds together, we have

$$\langle \phi_0 | A_X B_Y | \phi_0 \rangle - \langle \phi_0 | A_X | \phi_0 \rangle \langle \phi_0 | B_Y | \phi_0 \rangle$$

$$\leq O\left( \|A\| \, \|B\| \min(|X|, |Y|) e^{-\tilde{\mu} d(X,Y)} \right)$$
$$- \left( \cancel{\langle \phi_0 | A_X B_Y | \phi_0 \rangle} + O\left( \|A\| \cdot \|B\| \, e^{-\Delta^2/4\alpha} \right) \right.$$
$$\left. - \cancel{\langle \phi_0 | A_X B_Y | \phi_0 \rangle} \right)$$

$$= O\left( \|A\| \, \|B\| \min(|X|, |Y|) \, e^{-\tilde{\mu} d(X,Y)} \right)$$

recalling that we earlier chose

$$\alpha = \frac{\Delta}{2cd} \quad \& \quad c = \frac{\mu}{2ks + \Delta/2} \qquad \square$$

## Lemma

For any $\alpha > 0$, $E \in \mathbb{R}$,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{\Theta}(-t)e^{-iEt}e^{-\alpha t^2}dt = \begin{cases} 1 + O(e^{-\Delta^2/4\alpha}) & E \geq \Delta \\ O(e^{-\Delta^2/4\alpha}) & E \leq -\Delta \end{cases}$$

where $\hat{\Theta}(t) = \sqrt{\frac{\pi}{2}}\left(\delta(t) + \frac{i}{\pi t}\right)$ is the Fourier transform of the step function

$$\Theta(\omega) = \begin{cases} 1 & \omega \leq 0 \\ 0 & \omega > 0 . \end{cases}$$

## Proof

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\ e^{-iEt}\ \hat{\Theta}(-t)\ \underbrace{e^{-\alpha t^2}}_{G_\alpha(t)}\ dt \qquad = \mathcal{F}[\hat{\Theta}(-t)\,G_\alpha(t)]$$

$$\overset{\shortparallel}{\hat{\Theta}(-t)\,G_\alpha(-t)}$$

$$= \frac{1}{\sqrt{2\pi}}[\Theta * \hat{G}_\alpha](-E) \qquad \mathcal{F}[\hat{f}(t)] = f(-\omega)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\ \Theta(-E-\omega)\ \frac{1}{\sqrt{2\alpha}}e^{-\omega^2/4\alpha}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\ \Theta(\omega-E)\ \frac{1}{\sqrt{2\alpha}}e^{-\omega^2/4\alpha} \qquad \begin{matrix} \omega \to -\omega \\ d\omega \to -d\omega \\ \pm\infty \to \mp\infty \end{matrix}$$

$$= \frac{1}{2\sqrt{\alpha\pi}}\int_{-\infty}^{E} d\omega\ e^{-\omega^2/4\alpha}$$

$$
= \begin{cases} \frac{1}{2\sqrt{\alpha\pi}} \left( \int_{-\infty}^{\infty} d\omega \, e^{-\omega^2/4\alpha} - \int_{E}^{\infty} d\omega \, e^{-\omega^2/4\alpha} \right) & E \geqslant \Delta \\ \frac{1}{2\sqrt{\alpha\pi}} \int_{-\infty}^{E} d\omega \, e^{-\omega^2/4\alpha} & E \leq -\Delta \end{cases}
$$

$$
= \begin{cases} 1 + O(e^{-\Delta^2/4\alpha}) & E \geqslant \Delta \\ O(e^{-\Delta^2/4\alpha}) & E \leq -\Delta \end{cases}
$$

using $\int_{E}^{\infty} d\omega \, e^{-\omega^2/4\alpha} \leq \int_{E}^{\infty} d\omega \, \frac{\omega}{\Delta} \, e^{-\omega^2/4\alpha} \qquad E \geq \Delta$

$$
\leq \frac{2\alpha}{\Delta} \, e^{-\Delta^2/4\alpha}
$$

$\square$

# Notes

## FT of $\Theta(\omega)$:

$$\hat{\Theta}(t) = \int_{-\infty}^{\infty} e^{-i\omega t} \, \Theta(\omega) \, d\omega$$

$$= \int_{-\infty}^{\infty} e^{-i\omega t} \, H(-\omega) \, d\omega$$

usual Heaviside step func$^{\underline{n}}$

$$= -\int_{\infty}^{-\infty} e^{i\omega t} \, H(\omega) \, d\omega$$

$\omega \rightarrow -\omega$
$d\omega \rightarrow -d\omega$
$\pm\infty \rightarrow \mp\infty$

$$= \int_{-\infty}^{\infty} e^{-i\omega(-t)} \, H(\omega) \, d\omega$$

$$= \hat{H}(-t) = \frac{1}{2}\left(\delta(-t) - \frac{i}{\pi(-t)}\right)$$

$$= \frac{1}{2}\left(\delta(t) + \frac{i}{\pi t}\right)$$

## FT of -ve argument:

$$\mathcal{F}[f(-t)] = \int_{-\infty}^{\infty} e^{-i\omega t} \, f(-t) \, dt$$

$$= -\int_{\infty}^{-\infty} e^{i\omega t} \, f(t) \, dt$$

$t \rightarrow -t$
$dt \rightarrow -dt$
$\pm\infty \rightarrow \mp\infty$

$$= \int_{-\infty}^{\infty} e^{-i(-\omega)t} \, f(t) \, dt$$

$$= \hat{f}(-\omega)$$