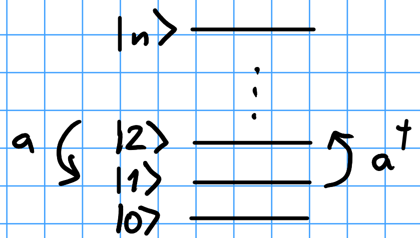
II. Representations of Angular Momentum Algebra

Our aim now is to find representations of the algebra of angular momentum operators (i.e. want to find matrix representations of J_3, J^2, $J_\pm$ etc, e.g. by finding their eigenvectors and eigenvalues.)

Recall harmonic oscillator



$$[a, a^\dagger] = 2m\hbar\omega$$
$$a|n\rangle = \sqrt{2m\hbar\omega n}\,|n-1\rangle$$
$$a^\dagger|n\rangle = \sqrt{2m\hbar\omega(n+1)}\;|n+1\rangle$$

This suggests the idea of introducing new operators:

Def$^{\underline{n}}$

"Ladder operators" J_+ and J_- are defined by

$$J_+ = J_1 + i J_2$$
$$J_- = J_1 - i J_2$$

Note: since they are observables, we need J_i to be self-adjoint $J_i^\dagger = J_i$, so

$$J_+^\dagger = (J_1 + iJ_2)^\dagger = J_1^\dagger - iJ_2^\dagger = J_1 - iJ_2 = J_-$$

Proposition 4

(i) $[J^2, J_\pm] = 0$

(ii) $J_\pm J_\mp = J^2 - J_3^2 \pm \hbar J_3$

(iii) $[J_+, J_-] = 2\hbar J_3$

(iv) $[J_3, J_\pm] = \pm \hbar J_\pm$

Proof

(i) Immediate from $[J^2, J_i] = 0$ (Proposition 3)

(ii)
$$\begin{aligned} J_+ J_- &= (J_1 + iJ_2)(J_1 - iJ_2) \\ &= J_1^2 + J_2^2 - iJ_1J_2 + iJ_2J_1 \\ &= J^2 - J_3^2 - i[J_1, J_2] \\ &= J^2 - J_3^2 + \hbar J_3 \end{aligned}$$

(iii) + (iv) Exercise. $\square$

Before persuing general method, let's first see a sneak preview of one of the simplest (but most important!) representations.

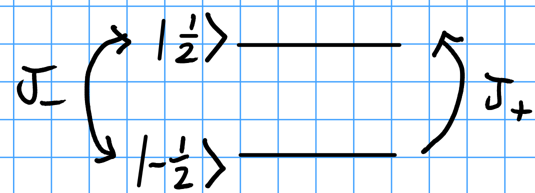
Example

We can guess a simple 2-dimensional representation by inspection (we will derive it again later more formally):

Consider two states $|+\frac{1}{2}\rangle$, $|-\frac{1}{2}\rangle$ defined to be eigenstates of J_3:

$$J_3 \, |+\tfrac{1}{2}\rangle = \frac{\hbar}{2} \, |\tfrac{1}{2}\rangle$$

$$J_3 \, |-\tfrac{1}{2}\rangle = -\frac{\hbar}{2} \, |-\tfrac{1}{2}\rangle$$



And let

$$J_+ \, |+\tfrac{1}{2}\rangle = 0$$

$$J_+ \, |-\tfrac{1}{2}\rangle = \hbar \, |+\tfrac{1}{2}\rangle$$

$$J_- \, |-\tfrac{1}{2}\rangle = 0$$

$$J_- \, |+\tfrac{1}{2}\rangle = \hbar \, |-\tfrac{1}{2}\rangle$$

If this is to give a representation, must verify that it's consistent with the algebra

e.g. $[J_+, J_-] = 2\hbar J_3$

$$\begin{aligned} [J_+, J_-] \,|{+\tfrac{1}{2}}\rangle &= (J_+ J_- - J_- J_+) \,|{+\tfrac{1}{2}}\rangle \\ &= J_+ \hbar \,|{-\tfrac{1}{2}}\rangle - 0 \\ &= \hbar^2 \,|{+\tfrac{1}{2}}\rangle \\ &= 2\hbar J_3 \,|{+\tfrac{1}{2}}\rangle \quad \text{as required} \end{aligned}$$

(Since $J_\pm$ generate the algebra via their commutation relation, along with similar result for $|{-\tfrac{1}{2}}\rangle$ this is in fact already sufficient to show these definitions are consistent with <u>entire</u> algebra, so do indeed define a representation.)

Exercise: verify
- $[J_+, J_-]\,|{-\tfrac{1}{2}}\rangle = 2\hbar J_3 \,|{-\tfrac{1}{2}}\rangle$,
- $|{\pm\tfrac{1}{2}}\rangle$ are eigenvectors of J^2,
- etc.

From above, deduce matrix elements $\langle \pm\tfrac{1}{2} | J_3 | \pm\tfrac{1}{2}\rangle$ etc. in $|{\pm\tfrac{1}{2}}\rangle$ basis:

$$J_3 = \begin{array}{c} \\ \langle +\frac{1}{2}| \\ \langle -\frac{1}{2}| \end{array} \begin{array}{c} \begin{array}{cc} |{+\frac{1}{2}}\rangle & |{-\frac{1}{2}}\rangle \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{array} \qquad J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

We now turn to the general derivation of all representations.

Corollary 5

Let $|\psi\rangle$ be a common eigenvector of J_3 and J^2 satisfying

$$J^2|\psi\rangle = \lambda\hbar^2|\psi\rangle \qquad J_3|\psi\rangle = m\hbar|\psi\rangle$$

(note: <u>no</u> assumptions on λ or m at this stage)

Then

(i) $J^2 J_\pm |\psi\rangle = \lambda\hbar^2 J_\pm|\psi\rangle$

(ii) $J_3 J_\pm |\psi\rangle = (m\pm 1)\hbar J_\pm|\psi\rangle$

(iii) $\| J_\pm |\psi\rangle\|^2 = [\lambda - m(m\pm 1)]\hbar^2 \, \||\psi\rangle\|^2$

(iv) $\lambda \geq m(m\pm 1)$ and $\lambda = m(m\pm 1)$ iff $J_\pm|\psi\rangle = 0$

Proof

(i) Follows immediately from $[J^2, J_\pm] = 0$ (Proposition 4 (i))

(ii) Using $[J_3, J_\pm] = \pm\hbar J_\pm$ (Proposition 4(iv)),

$$\begin{aligned} J_3 J_\pm|\psi\rangle &= (J_\pm J_3 \pm \hbar J_\pm)|\psi\rangle \\ &= (J_\pm m\hbar \pm \hbar J_\pm)|\psi\rangle \\ &= (m\pm 1)\hbar J_\pm|\psi\rangle \end{aligned}$$

(iii) $\| J_\pm |\psi\rangle \|^2 = \langle\psi| J_\pm^\dagger J_\pm |\psi\rangle$

$= \langle\psi| J_\mp J_\pm |\psi\rangle$

$= \langle\psi| J^2 - J_3^2 \mp \hbar J_3 |\psi\rangle$

(Proposition 4 (ii))

$= (\lambda \hbar^2 - m^2 \hbar^2 - m \hbar^2) \langle\psi|\psi\rangle$

$= [\lambda - m(m \pm 1)] \hbar^2 \, \| |\psi\rangle \|^2$

(iv) $\| J_\pm |\psi\rangle \|^2 \geqslant 0$ with equality iff $J_\pm |\psi\rangle = 0$

Thus from (iii)

$\lambda \geqslant m(m \pm 1)$

with equality iff $J_\pm |\psi\rangle = 0$ $\square$

Similar situation to harmonic oscillator – given λ, m bounded from below, but unlike harm. osc. m also bounded from above.

Theorem 6

Eigenvalues of J^2 have form $j(j+1)\hbar^2$ for $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots, \frac{n}{2}$

For each value of j, eigenvalues of J_3 have form $m\hbar$ for $m = -j, -j+1, \ldots, j-1, j$ and each eigenvalue has same degeneracy.

Proof

Rewriting $\lambda \geq m(m+1)$ (Corollary 5 (iv)) as

$$\lambda + \frac{1}{4} \geq \left(m \pm \frac{1}{2}\right)^2$$

it is clear that, for given λ, m is bounded above and below.

Let $|\psi\rangle$ again be a simultaneous eigenvector of J^2, J_3 with eigvals. $\lambda\hbar^2$ and $m\hbar$ respectively.

From $J_3 J_\pm |\psi\rangle = (m \pm 1)\hbar |\psi\rangle$ (Corollary 5 (ii)) we have that

$J_+^N |\psi\rangle$ is eigvect. of J_3 with eigval. $(m+N)\hbar$

$J_-^N |\psi\rangle$ " " " J_3 " " $(m-N)\hbar$,

or they vanish.

Since m is bounded, $J_{\pm}^{N+1}|\psi\rangle$ <u>must</u> eventually vanish for large enough N.

Let p, q be <u>minimum</u> such (+ve) integers, i.e.

$$J_+^{p+1}|\psi\rangle = 0 \quad \text{but} \quad J_+^{p}|\psi\rangle \neq 0$$
$$J_-^{q+1}|\psi\rangle = 0 \qquad\qquad J_-^{q}|\psi\rangle \neq 0.$$

Since $J_+^p|\psi\rangle$ is eigvect. of J_3 with eigval. $(m+p)\hbar$ and $J_+(J_+^p|\psi\rangle) = 0$, must have

$$\lambda = (m+p)(m+p+1)$$

by Corollary 5 (iv).

Similarly since $J_-^q|\psi\rangle$ has eigval. $(m-q)\hbar$ and $J_-(J_-^q|\psi\rangle) = 0$, must have

$$\lambda = (m-q)(m-q-1)$$

Thus

$$(m+p)(m+p+1) = (m-q)(m-q-1)$$

so

$$\cancel{m^2} + 2mp + p^2 + m + p = \cancel{m^2} - 2mq + q^2 - m + q$$

$$2m(p+q+1) = (q-p)(p+q+1)$$

Thus, since $p+q+1 > 0$,

$$m = \frac{q-p}{2}$$

Let $j = m+p = \frac{p+q}{2}$.

Then

$$\lambda = (m+p)(m+p+1) = j(j+1) \quad [*]$$

Summarising,

$m = \frac{q-p}{2}$ $\Rightarrow$ m is integer or half-integer

$m = j-p$ $\Rightarrow$ m differs from j by integer

$j = \frac{p+q}{2}$ $\Rightarrow$ j is integer or half-integer

$\lambda = j(j+1)$ as required (from [*]).

Remains to show that all values of m in range $-j, -j+1, \ldots, j-1, j$ (and no others) do in fact occur.

First, note $|\psi_j\rangle = J_+^p |\psi\rangle$ has J_3 eig. val $(m+p)\hbar = j\hbar$, and $J_+|\psi_j\rangle = 0$ so this is maximum value of m.

Then $J_-^{j-m}|\psi_j\rangle$ has J_3 eig. val $(j-(j-m))\hbar = m\hbar$ so all values $j, j-1, j-2 \ldots$ do occur.

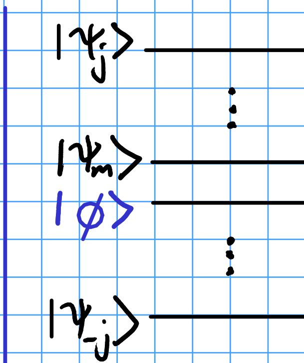
Finally, $J_-^q|\psi\rangle$ has J_3 eig. val $(m-q)\hbar = \left(\frac{q-p}{2} - q\right)\hbar = -\frac{q+p}{2}\hbar = -j\hbar$

and $J_- J_-^q |\psi\rangle = 0$ so this is minimum value of m.

Degeneracies

Claim in Theorem is trivially true for $j = 0$, since there is only one possible value $m = 0$.
From now on, assume $j > 0$.

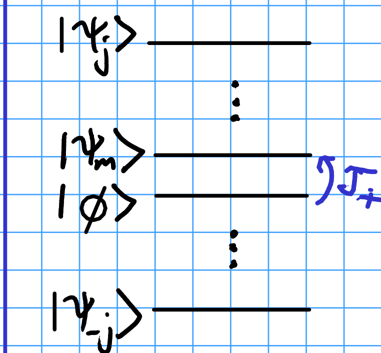
Assume $|\psi_m\rangle$ is unique state s.t. $J_3 |\psi_m\rangle = m\hbar |\psi_m\rangle$, $m > -j$.



Let $|\phi\rangle$ be any state satisfying $J_3 |\phi\rangle = (m-1)\hbar |\phi\rangle$, and consider $J_+ |\phi\rangle$.

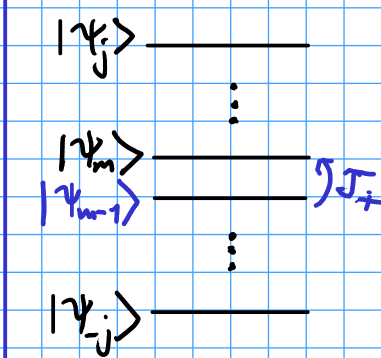
We have

$$\begin{aligned} J_3 J_+ |\phi\rangle &= (J_+ J_3 + \hbar J_+) |\phi\rangle \\ &= (J_+ (m-1)\hbar + \hbar J_+) |\phi\rangle \\ &= m\hbar J_+ |\phi\rangle \end{aligned}$$



Thus $J_+ |\phi\rangle = \alpha |\psi_m\rangle$ since by assumption $|\psi_m\rangle$ is unique state with eigval. $m\hbar$.

Similarly, for $|\psi_{m-1}\rangle = J_- |\psi_m\rangle$
$J_+ |\psi_{m-1}\rangle = \beta |\psi_m\rangle$.



Now

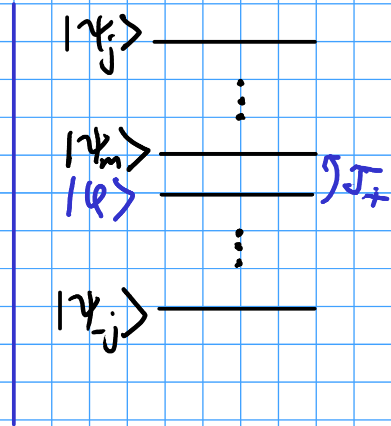
$$\begin{aligned} J_3 (\underbrace{\beta |\phi\rangle - \alpha |\psi_{m-1}\rangle}_{|\varphi\rangle}) &= (m-1)\hbar (\beta |\phi\rangle - \alpha |\psi_{m-1}\rangle) \\ &= (m-1)\hbar |\varphi\rangle \end{aligned}$$

so $|\varphi\rangle$ is also an eigvect. of J_3 with eigval. $(m-1)\hbar < m\hbar \le j\hbar$

Thus by Corollary 5(iv),

$J_+ |\varphi\rangle \neq 0$ [*]

unless trivially $|\varphi\rangle = 0$.



But

$$J_+ |\varphi\rangle = J_+ (\beta |\phi\rangle - \alpha |\psi_{m-1}\rangle)$$
$$= \beta\alpha |\psi_m\rangle - \alpha\beta |\psi_m\rangle = 0$$

which contradicts [*] unless $|\varphi\rangle = \beta|\phi\rangle - \alpha|\psi_{m-1}\rangle = 0$.

Thus $|\phi\rangle = \frac{\alpha}{\beta} |\psi_{m-1}\rangle$

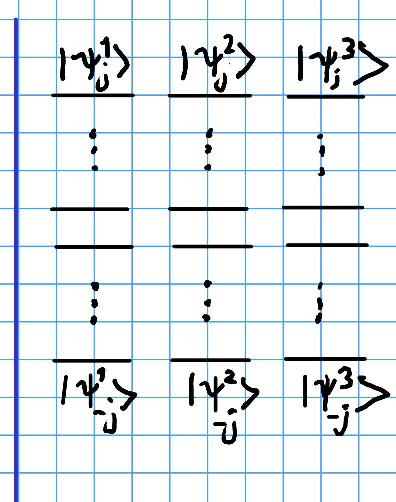
i.e. any $|\phi\rangle$ satisfying $J_3 |\phi\rangle = (m-1)\hbar |\phi\rangle$ must be proportional to $|\psi_{m-1}\rangle$.

Therefore, assuming $|\psi_m\rangle$ is unique, $|\psi_{m-1}\rangle$ is the unique (up to normalisation) eigvect of J_3 with eig.val $(m-1)\hbar$.

By induction, if there exists unique $|\psi_j\rangle$ for eigval. $j\hbar$, then each eigval. $m\hbar$ has unique eigvect. $J_-^{\,j-m} |\psi_j\rangle$.

What if $j\hbar$ is degenerate ?

Can repeat argument for each $|\psi_j^k\rangle$ spanning degenerate subspace of eigvects for eigval. $j\hbar$ to show that each eigval $m\hbar$ must have same degeneracy as that of $j\hbar$.



Corollary 7

If there are no degeneracies, eigenspace on which J^2 takes eigenvalue $j(j+1)\hbar^2$ has dimension $2j+1$, spanned by basis $\{ |\psi_m\rangle : 0 \leq j-m \leq 2j \}$ such that

$$J_3 |\psi_m\rangle = m\hbar |\psi_m\rangle$$
$$J_\pm |\psi_m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\, \hbar\, |\psi_{m\pm 1}\rangle$$

Proof

Since $J_+|\psi_m\rangle$ is J_3 eigvect. with eigval. $(m+1)\hbar$, have $J_+|\psi_m\rangle = c\,|\psi_{m+1}\rangle$.

Can choose phase of $|\psi_{m+1}\rangle$ s.t. $c \in \mathbb{R}$.

Then

$$|c|^2 \langle \psi_{m+1} | \psi_{m+1} \rangle = \langle \psi_m | J_- J_+ | \psi_m \rangle$$
$$= \langle \psi_m | J^2 - J_3^2 - \hbar J_3 | \psi_m \rangle$$
$$= j(j+1)\hbar^2 - m^2\hbar^2 - m\hbar^2$$

so

$$c = \sqrt{j(j+1) - m(m+1)}\, \hbar$$
$$= \sqrt{(j-m)(j+m+1)}\, \hbar \quad \text{as required}$$

$J_-|\psi_m\rangle$ left as exercise. $\square$

Representations of Angular Momentum

Theorem 6 and Corollary 7 provide the tools we need to construct representations for each permitted value of j.

Example: $j = 0$

$2j+1 = 1$ one-dimensional (by Corollary 7)

$J_3 |\psi_0\rangle = 0 \qquad J^2 |\psi_0\rangle = 0 \qquad J_\pm |\psi_0\rangle = 0$

"trivial representation"

Example: $j = \frac{1}{2}$ "spin $\frac{1}{2}$"

$2j+1 = 2$ two-dimensional

space spanned by $|\frac{1}{2}, \frac{1}{2}\rangle$ and $|\frac{1}{2}, -\frac{1}{2}\rangle$

↳ write as $|\pm\frac{1}{2}\rangle$

$$J_3 |\pm\tfrac{1}{2}\rangle = \pm\frac{\hbar}{2} |\pm\tfrac{1}{2}\rangle$$

$$J_+ |+\tfrac{1}{2}\rangle = 0 = J_- |-\tfrac{1}{2}\rangle$$

$$J_+ |-\tfrac{1}{2}\rangle = \sqrt{(\tfrac{1}{2}+\tfrac{1}{2})(\tfrac{1}{2}+(-\tfrac{1}{2})+1)}\,\hbar\, |+\tfrac{1}{2}\rangle = \hbar |+\tfrac{1}{2}\rangle$$

$$J_- |+\tfrac{1}{2}\rangle = \hbar |-\tfrac{1}{2}\rangle$$

Exercise: verify $J_\pm$, J_3 do indeed obey correct algebra when acting on $|\pm\frac{1}{2}\rangle$ states.

With respect to $|\pm\frac{1}{2}\rangle$ basis

$$J_3 = \begin{pmatrix} \langle\frac{1}{2}|J_3|\frac{1}{2}\rangle & \langle\frac{1}{2}|J_3|-\frac{1}{2}\rangle \\ \langle-\frac{1}{2}|J_3|\frac{1}{2}\rangle & \langle-\frac{1}{2}|J_3|-\frac{1}{2}\rangle \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$J_+ = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad J_- = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$J_1 = \frac{1}{2}(J_+ + J_-) = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$J_2 = \frac{1}{2i}(J_+ - J_-) = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

This is the representation we constructed previously by inspection.

Note matrices obey same algebra as operators, as required, e.g.

$$J_1 J_2 = \frac{\hbar^2}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\hbar^2}{4} i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[J_1, J_2] = \frac{\hbar^2}{2} i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\hbar\,\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\hbar J_3 \checkmark$$

Called the "spin representation" of angular momentum. The matrices in this representation are important enough to have a name:

Def

"Pauli spin matrices" or just "Pauli matrices":

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Example: $j=1$

$2j+1=3$ 3-dimensional

basis $|+1\rangle$, $|0\rangle$, $|-1\rangle$

$$J_3\,|+1\rangle = \hbar\,|+1\rangle$$
$$J_3\,|0\rangle = 0$$
$$J_3\,|-1\rangle = -\hbar\,|-1\rangle$$

$$J_-\,|+1\rangle = \sqrt{2}\,\hbar\,|0\rangle$$
$$J_-\,|0\rangle = \sqrt{2}\,\hbar\,|-1\rangle$$
$$J_-\,|-1\rangle = 0$$

$$J_+\,|+1\rangle = 0$$
$$J_+\,|0\rangle = \sqrt{2}\,\hbar\,|+1\rangle$$
$$J_+\,|-1\rangle = \sqrt{2}\,\hbar\,|0\rangle$$

w.r.t. this basis, the ang. mom. operators are represented by the matrices

$$J_+ = \hbar\begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \qquad J_- = \hbar\begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$J_3 = \hbar\begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$$