II. Multiple Particles & Tensor Products

Aim to describe space of states of two (or more) systems with Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, and also to describe operators acting on this space.

Amongst these states, expect to find states in which system A is in state $|\psi_A\rangle$, and system B in state $|\psi_B\rangle$.

However, as we will soon see, <u>linearity</u> of Q.M. implies not all states of combined system are of this form

$\rightarrow$ entanglement, perhaps <u>the</u> strangest & most remarkable feature of Q.M.

Tensor product state spaces

Imagine two systems A and B that are spatially separated

A $\mathcal{H}_A$ basis $\{|a_i\rangle\}$ B $\mathcal{H}_B$ basis $\{|b_i\rangle\}$

Introduce <u>tensor product</u> of $\mathcal{H}_A$ and $\mathcal{H}_B$, denoted $\mathcal{H}_A \otimes \mathcal{H}_B$, to describe state space of joint system.

One way to describe it is as vector space spanned by the basis vectors

$$|a_i\rangle \otimes |b_j\rangle$$

Arbitrary vector in $\mathcal{H}_A \otimes \mathcal{H}_B$ is then

$$\sum_i \alpha_{ij} \; |a_i\rangle \otimes |b_j\rangle$$

[Note: we have not proven that this gives same space regardless of which bases we choose for $\mathcal{H}_A$ and $\mathcal{H}_B$.
To make this mathematically rigorous, would need to start from basis independent definition of tensor product, and prove above construction is unique (up to isomorphism). We will not do this, but see e.g. Hannabus 16.5.]

Example: $\mathbb{C}^2 \otimes \mathbb{C}^2$

basis $\{|0\rangle, |1\rangle\}$ basis $\{|0\rangle, |1\rangle\}$

↳ 4 basis elements:

$$|0\rangle \otimes |0\rangle \xrightarrow{\text{shorthand}} |0\rangle|0\rangle$$

$$|0\rangle \otimes |1\rangle \qquad |0\rangle|1\rangle$$

$$|1\rangle \otimes |0\rangle \qquad |1\rangle|0\rangle$$

$$|1\rangle \otimes |1\rangle \qquad |1\rangle|1\rangle$$

Arbitrary vector is

$$\sum_{i,j=0}^{1} \alpha_{ij} |i\rangle|j\rangle$$

$$= \alpha_{00} |0\rangle|0\rangle + \alpha_{01} |0\rangle|1\rangle + \alpha_{10} |1\rangle|0\rangle + \alpha_{11} |1\rangle|1\rangle$$

$\rightarrow$ $\mathbb{C}^2 \otimes \mathbb{C}^2$ is (complex) vector space with 4 basis elements, thus $\mathbb{C}^2 \otimes \mathbb{C}^2 \hat{=} \mathbb{C}^4$.

Tensor product is <u>linear</u> by definition:

If

$$|\psi_A\rangle = a_0 |0\rangle + a_1 |1\rangle$$

$$|\psi_B\rangle = b_0 |0\rangle + b_1 |1\rangle$$

then state of combined system is

$$|\psi_A\rangle \otimes |\psi_B\rangle = (a_0 |0\rangle + a_1 |1\rangle) \otimes (b_0 |0\rangle + b_1 |1\rangle)$$

$$= a_0 b_0 |0\rangle|0\rangle + a_0 b_1 |0\rangle|1\rangle + a_1 b_0 |1\rangle|0\rangle + a_1 b_1 |1\rangle|1\rangle$$

[Or, in vector notation,

$$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \otimes \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} = \text{``}\begin{pmatrix} a_0 \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} \\ a_1 \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} \end{pmatrix}\text{''} = \begin{pmatrix} a_0 b_0 \\ a_0 b_1 \\ a_1 b_0 \\ a_1 b_1 \end{pmatrix}$$

]

<u>However</u>, it is fundamentally important that not all states in $\mathcal{H}_A \otimes \mathcal{H}_B$ are of form

$|\psi_A\rangle \otimes |\psi_B\rangle$ called "product form".

E.g.

<u>Claim</u>

$$|\phi\rangle = \frac{|0\rangle|0\rangle + |1\rangle|1\rangle}{\sqrt{2}}$$ is not of product form.

<u>Proof</u>

If $|\phi\rangle$ was of product form, could be written

$$(a_0|0\rangle + a_1|1\rangle) \otimes (b_0|0\rangle + b_1|1\rangle)$$

$$= a_0 b_0 |0\rangle|0\rangle + a_0 b_1 |0\rangle|1\rangle + a_1 b_0 |1\rangle|0\rangle + a_1 b_1 |1\rangle|1\rangle \qquad (*)$$

for some a_0, a_1, b_0, b_1.

Now $|0\rangle|1\rangle$ term in $(*)$ implies $a_0 b_1 = 0$, i.e. $a_0 = 0$ or $b_1 = 0$.

But then $|0\rangle|0\rangle$ or $|1\rangle|1\rangle$ term would vanish!

$\longrightarrow$ contradiction.

Definition 1

Any state which cannot be written in product form is said to be <u>entangled</u>.

Careful!

$$\frac{|0\rangle|0\rangle + |1\rangle|1\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle}{2}$$

$$= \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)$$

so it <u>is</u> of product form.

It is a deep fact about the way nature appears to behave that spaces of many particles combine via a tensor product, implying the existence of entangled states.

Inner Products

Definition 2

Inner product on $\mathcal{H}_A \otimes \mathcal{H}_B$ defined in terms of inner products $\langle v_A | w_A \rangle$ on $\mathcal{H}_A$ and $\langle v_B | w_B \rangle$ on $\mathcal{H}_B$ by

$$(\langle v_A | \langle v_B |)(|w_A\rangle |w_B\rangle) = \langle v_A | w_A \rangle \langle v_B | w_B \rangle$$

Check $\langle \psi | \phi \rangle = \langle \phi | \psi \rangle^*$:

$$\left[(\langle v_A | \langle v_B |)(|w_A\rangle |w_B\rangle)\right]^*$$

$$= \left[\langle v_A | w_A \rangle \langle v_B | w_B \rangle\right]^*$$

$$= \langle v_A | w_A \rangle^* \langle v_B | w_B \rangle^*$$

$$= \langle w_A | v_A \rangle \langle w_B | v_B \rangle$$

$$= (\langle w_A | \langle w_B |)(|v_A\rangle |v_B\rangle) \quad \text{as required.}$$

Def. 2 is sufficient to calculate all inner products, via linearity, e.g.

$$(\langle v_A | \langle v_B |)(\alpha |w_A\rangle |w_B\rangle + \beta |z_A\rangle |z_B\rangle)$$

$$= \alpha \langle v_A | w_A \rangle \langle v_B | w_B \rangle + \beta \langle v_A | z_A \rangle \langle v_A | z_B \rangle$$

Example

$$(\langle 0|\langle 1|)(|0\rangle|0\rangle) = \langle 0|0\rangle\langle 1|0\rangle = 1\cdot 0 = 0$$

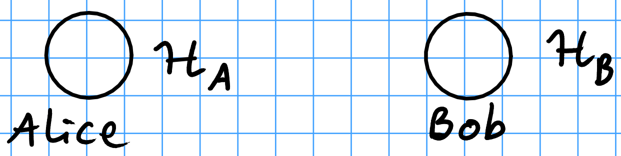
Example

Inner product of $|0\rangle|1\rangle + i|1\rangle|0\rangle$
with $|0\rangle|1\rangle + i|1\rangle|0\rangle$ is

$$(\langle 0|\langle 1| - i\langle 1|\langle 0|)(|0\rangle|1\rangle + i|1\rangle|0\rangle) = 0$$

note sign change
from Hermitian conjugate

Operators on Tensor Product Spaces



Operator A acts on $\mathcal{H}_A$

" B " " $\mathcal{H}_B$

Define linear operator $A \otimes B$ acting on $\mathcal{H}_A \otimes \mathcal{H}_B$:

- on basis $|a_i\rangle \otimes |b_j\rangle$

$$(A \otimes B)\, |a_i\rangle |b_j\rangle = A|a_i\rangle \otimes B|b_j\rangle$$

- on general state, extend by linearity:

$$(A \otimes B) \sum_{ij} \alpha_{ij} |a_i\rangle |b_j\rangle = \sum_{ij} \alpha_{ij} A|a_i\rangle \otimes B|b_j\rangle$$

This corresponds to Alice applying operator A to her system & Bob applying operator B to his system.

Example

Recall Pauli operators on $\mathbb{C}^2$

$X|0\rangle = |1\rangle$ $\qquad$ $Z|0\rangle = |0\rangle$

$X|1\rangle = |0\rangle$ $\qquad$ $Z|1\rangle = -|1\rangle$

Can define $X \otimes Z$ on $\mathbb{C}^2 \otimes \mathbb{C}^2$, e.g.

$$X \otimes Z\, |0\rangle|0\rangle = X|0\rangle \otimes Z|0\rangle = |1\rangle|0\rangle$$

$$X \otimes Z\, |1\rangle|1\rangle = -|0\rangle|1\rangle$$

etc.

Thus

$$X \otimes Z \left(\frac{|0\rangle|0\rangle + |1\rangle|1\rangle}{\sqrt{2}} \right) = \frac{|1\rangle|0\rangle - |0\rangle|1\rangle}{\sqrt{2}}$$

Note:

$X \otimes Z \neq Z \otimes X$, e.g.

$$Z \otimes X \, |0\rangle|0\rangle = \underbrace{|0\rangle|1\rangle \neq |1\rangle|0\rangle}_{\text{orthogonal!}} = X \otimes Z \, |0\rangle|0\rangle$$

Example

Operator corresponding to doing nothing is identity operator $\mathbb{1}$:

$$\mathbb{1} \, |\psi\rangle = |\psi\rangle \quad \forall \, |\psi\rangle$$

Thus operator corresponding to Alice doing X and Bob doing nothing is $X \otimes \mathbb{1}$:

$$X \otimes \mathbb{1} \, |0\rangle|0\rangle = |1\rangle|0\rangle$$

$$X \otimes \mathbb{1} \left(\frac{|0\rangle|0\rangle + |1\rangle|1\rangle}{\sqrt{2}} \right) = \frac{|1\rangle|0\rangle + |0\rangle|1\rangle}{\sqrt{2}}$$

etc.

Proposition 3

If operator A has eigvals. λ_i and eigvects $|a_i\rangle$
B " " μ_i " " $|b_i\rangle$

then $A \otimes B$ has eigvals $\lambda_i \mu_j$ and eigvects. $|a_i\rangle |b_j\rangle$

Proof

$$A \otimes B \; |a_i\rangle |b_j\rangle = A|a_i\rangle \otimes B|b_j\rangle$$
$$= \lambda_i |a_i\rangle \otimes \mu_j |b_j\rangle$$
$$= \lambda_i \mu_j \; |a_i\rangle |b_j\rangle$$

Note that this implies $A \otimes B$ has spectral decomposition

$$A \otimes B = \sum_{ij} \lambda_i \mu_j \; |a_i\rangle |b_j\rangle \langle a_i| \langle b_j|$$
$$= \left(\sum_i \lambda_i |a_i\rangle\langle a_i| \right) \otimes \left(\sum_j \mu_j |b_j\rangle\langle b_j| \right)$$

as expected

Alternatively, as matrices we have e.g.

$$A = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \qquad B = \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix}$$

$$A \otimes B = \text{``}\begin{pmatrix} a_{00}\begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{10} \end{pmatrix} & a_{01}\begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix} \\ a_{10}\begin{pmatrix} & \text{''} & \end{pmatrix} & a_{11}\begin{pmatrix} & \text{''} & \end{pmatrix} \end{pmatrix}\text{''}$$

$$= \begin{pmatrix} a_{00}b_{00} & a_{00}b_{01} & a_{01}\,b_{00} & a_{01}\,b_{01} \\ a_{00}\,b_{10} & a_{00}b_{11} & & \\ a_{10}\,b_{00} & a_{10}\,b_{01} & \text{etc.} & \\ a_{10}\,b_{10} & a_{10}\,b_{11} & & \end{pmatrix}$$

Proposition 4

$$(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$$

Proof

Show this on a basis — extends by linearity

$$\langle a_i | \langle b_j | \, A^\dagger \otimes B^\dagger \, | a_k \rangle | b_\ell \rangle$$

$$= \langle a_i | A^\dagger | a_k \rangle \langle b_j | B^\dagger | b_\ell \rangle$$

$$= \left(\langle a_k | A | a_i \rangle \langle b_\ell | B | b_j \rangle \right)^*$$

$$= \left(\langle a_k | \langle b_\ell | \, A \otimes B \, | a_i \rangle | b_j \rangle \right)^*$$

$$= \langle a_i | \langle b_j | (A \otimes B)^\dagger | a_k \rangle | b_\ell \rangle$$

Proposition 5

$$(A \otimes B)(C \otimes D) = AC \otimes BD$$

Proof

On basis,

$$(A \otimes B)(C \otimes D)\,|a_i\rangle\,|b_j\rangle$$

$$= (A \otimes B)\big(C\,|a_i\rangle \otimes D\,|b_j\rangle\big)$$

$$= AC\,|a_i\rangle \otimes BD\,|b_j\rangle$$

$$= (AC \otimes BD)\,|a_i\rangle\,|b_j\rangle.$$

By linearity, extends to entire space $\mathcal{H}_A \otimes \mathcal{H}_B$.

Note: have discussed tensor products of 2 spaces, but everything extends in obvious way to 3 and more spaces (see Question Sheet).

Measurement Again

Consider measuring Z in state $|\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$

$$Z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

$P_0 = |0\rangle\langle 0|$ is projector onto eigenstate corresponding to eigval. $+1$.

$P_1 = |1\rangle\langle 1|$ is projector onto eigstate corresponding to eig val -1.

Post - measurement state given outcome $+1$ is

$$\frac{P_0 |\psi\rangle}{\| P_0 |\psi\rangle \|} = \frac{|0\rangle\langle 0| \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right)}{\| P_0 |\psi\rangle \|} = \frac{\frac{1}{\sqrt{2}} |0\rangle}{\| \frac{1}{\sqrt{2}} |0\rangle \|}$$

$$= \frac{\frac{1}{\sqrt{2}} |0\rangle}{1/\sqrt{2}} = |0\rangle \quad \text{as expected}$$

Probability of outcome is

$$\langle \psi | P_0 | \psi \rangle = \langle \psi | \left(\tfrac{1}{\sqrt{2}} |0\rangle \right) = \tfrac{1}{\sqrt{2}} \langle \psi | 0 \rangle$$

$$= \tfrac{1}{\sqrt{2}} \cdot \tfrac{1}{\sqrt{2}} = \tfrac{1}{2} \quad \text{as expected}$$

Similarly, for -1 outcome:

$$\frac{P_1 |\psi\rangle}{\| P_1 |\psi\rangle \|} = |1\rangle , \qquad \langle \psi | P_1 | \psi \rangle = \tfrac{1}{2}$$

as expected.

This formulation allows us to extend out description of measurement in Q.M. to e.g. measuring $Z \otimes \mathbb{1}$ on state

$$|\psi\rangle = \frac{|0\rangle|0\rangle + |1\rangle|1\rangle}{\sqrt{2}}$$

Z has eigvals $+1, -1$

$\mathbb{1}$ " " $+1$ $(\times 2)$

$Z \otimes \mathbb{1}$ " " $+1 (\times 2), -1 (\times 2)$ (Proposition 3)

$$\begin{aligned} Z \otimes \mathbb{1} &= (|0\rangle\langle 0| - |1\rangle\langle 1|) \otimes \mathbb{1} \\ &= \underbrace{|0\rangle\langle 0| \otimes \mathbb{1}}_{P_0} - \underbrace{|1\rangle\langle 1| \otimes \mathbb{1}}_{P_1} \end{aligned}$$

projector corresponding to eigval. +1

projector corresponding to eigval. −1

Post-measurement state given outcome +1 is

$$\frac{P_0 |\psi\rangle}{\| P_0 |\psi\rangle \|} = \frac{(|0\rangle\langle 0| \otimes \mathbb{1}) \frac{1}{\sqrt{2}} (|0\rangle|0\rangle + |1\rangle|1\rangle)}{\| P_0 |\psi\rangle \|}$$

$$= \frac{\frac{1}{\sqrt{2}} |0\rangle|0\rangle}{\| \frac{1}{\sqrt{2}} |0\rangle|0\rangle \|} = |0\rangle|0\rangle$$

Probability of outcome +1 is

$$\begin{aligned} \langle \psi | P_0 | \psi \rangle &= \langle \psi | (|0\rangle\langle 0| \otimes \mathbb{1}) \tfrac{1}{\sqrt{2}} (|0\rangle|0\rangle + |1\rangle|1\rangle) \\ &= \tfrac{1}{2} (\langle 0|\langle 0| + \langle 1|\langle 1|) |0\rangle|0\rangle = \tfrac{1}{2}. \end{aligned}$$

Definition 6

Let M be a Hermitian measurement operator with (possibly degenerate) eigenvalues λ_i.
Let

$$P_i = \sum_k |e_i^{(k)}\rangle\langle e_i^{(k)}|$$

be the projector onto the space of eigenstates $|e_i^{(k)}\rangle$ with eigenvalue λ_i.

Then, if we measure M in state $|\psi\rangle$,

(i) the possible measurement outcomes are given by the eigenvalues λ_i;

(ii) the probability of obtaining outcome λ_i is $p(\lambda_i) = \langle\psi| P_i |\psi\rangle$;

(iii) the post-measurement state is $\dfrac{P_i|\psi\rangle}{\| P_i |\psi\rangle\|}$.

Corollary 7

Expectation value of measurement M in state $|\psi\rangle$ is $\langle M\rangle = \langle\psi| M |\psi\rangle$.

Proof

$$\langle M\rangle = \sum_i p(\lambda_i)\lambda_i = \sum_i \langle\psi| P_i |\psi\rangle \lambda_i$$

$$= \langle\psi| \sum_i \lambda_i P_i |\psi\rangle = \langle\psi| M |\psi\rangle.$$